Minimum Cost Spanning Tree Situations with Some Structures

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1 Introduction: minimum cost spanning tree situation (mcsts)

minimum cost spanning tree situation (mcsts) (N_0, C)

- $N = \{1, 2, \ldots, n\}$: set of agents who are willing to be connected as cheap as possible to a source
- 0 : source, supplier of a service
- $N_0 = \{0\} \cup N$
- $C = (c_{ij})_{i,j \in N_0}$: $(n+1) \times (n+1)$ cost matrix, $c_{ij} \geq 0$, $c_{ii} = 0$

$S \subset N \rightarrow$ induced mcsts (S_0, C)

network T over N_0 : subset of $\{(i,j) \mid i,j \in N_0, i \neq j\}$
each arc (i,j) is undirected, i.e., $(i,j) = (j,i) \rightarrow c_{ij} = c_{ji}$
T_S: network induced by T over S, i.e., $T_S = \{(i,j) \mid i,j \in S\}$

$i, j \in N_0$ are **connected** in T if there exists a sequence of arcs (called path) $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in T$ for all $h = 1, \ldots, l$, $i = i_0$ and $j = i_l$

(spanning) tree is a network where there is a unique path from i to 0 for all $i \in N$
tree $T = \{(i^0, i)\}$, i^0 is the first agent (or source) in the unique path in T from i to 0

cost of a network T

$$c(N_0, C, T) = \sum_{(i,j) \in T} c_{ij} = c(T)$$

minimum cost spanning tree (mt) for (N_0, C) is a tree T such that

$c(T) = \min\{c(T') \mid T' \text{ is a network s.t. } N_0 \text{ is connected in } T'\} = \min\{c(T') \mid T' \text{ is a tree}\}$

the cost associated with any mt T in (N_0, C) is denoted by $m(N_0, C)$
the minimum cost of an induced mcsts (S_0, C) is denoted by $m(S_0, C) \longrightarrow$ game $v_C(S)$

mcsts $(N_0, C) \longrightarrow$ mt $T \longrightarrow$ minimal network associated with T (Bird)
T_{ij} : unique path in T from i to j

$$c^*_{ij} = \max_{(k,l) \in T_{ij}} c_{kl}$$

this value does not depned on $T \longrightarrow$ irreducible form (N_0, C^*)

mcsts (N_0, C)

- **construction of a network of minimal cost** connecting all agents to the source
  - Prim algorithm 1957
  - Kruskal algorithm 1956
- **how to divide the cost** of connecting agents to the source among them (cost allocation rules)
  - Prim algoritm → Bird 1976, Dutta and Kar 2004
  - Kruskal algorithm → ERO rule (Feltkamp et al. 1994, Branzei et al. 2004)
    obligation rules (Tijs et al. 2006, Bergantiños and Kar 2008,
    Lorenzo and Lorenzo-Freier 2009)
    generalized obligation rules (Bergantiños et al. 2007)
    construct and charge rules (Moretti et al. 2007)
  - cooperative game → core, nucleolus (Granot and Hubereman 1994)
    Shapley value (Kar 2002)
  - irreducible form → cooperative game (Bird 1976) → Shapley value etc.
    optimistic game → weighted Shapley value (Bergantiños et al. 2007,8)

contruction of mt (1)

Prim algorithm

Stage 0: Let $S^0 = \{0\}$ and $T^0 = \emptyset$.

Stage 1: Take an arc $(0, i_1)$ such that $c_{0i_1} = \min_{j \in N} c_{0j}$. Let $S^1 = \{0, i_1\}$ and $T^1 = \{(0, i_1)\}$.

Stage $p+1$: Assume that we have defined $S^p \subset N_0$ and a tree T^p on N_0. Take an arc (i^0_{p+1}, i_{p+1}) with $i^0_{p+1} \in S^p$ and $i_{p+1} \in N_0 \setminus S^p$ such that $c_{i^0_{p+1} i_{p+1}} = \min_{k \in S^p, l \in N_0 \setminus S^p} c_{kl}$. Let $S^{p+1} = S^p \cup \{i_{p+1}\}$ and $T^{p+1} = T^p \cup \{(i^0_{p+1}, i_{p+1})\}$.

This process is completed in n stages.

contruction of mt (2)

Kruskal algorithm

Stage 0: Let $A^0 = \{(i,j) \mid i,j \in N_0, i \neq j\}$ and $T^0 = \emptyset$.

Stage 1: Take an arc $(i_1, j_1) \in A^0$ such that $c_{i_1 j_1} = \min\limits_{(k,l) \in A^0} c_{kl}$. Let $A^1 = A^0 \setminus \{(i_1 j_1)\}$ and $T^1 = \{(i_1, j_1)\}$.

Stage $p+1$: Assume that we have defined A^p and T^p. Take an arc $(i,j) \in A^p$ such that $c_{ij} = \min\limits_{(k,l) \in A^p} c_{kl}$. Two cases are possible:

1. $T^p \cup \{(i,j)\}$ has a cycle. Go to the beginning of Stage $p+1$ with $A^p = A^p \setminus \{(i,j)\}$ and T^p the same.
2. $T^p \cup \{(i,j)\}$ has no cycles. Take $(i_{p+1} j_{p+1}) = (i,j)$, $A^{p+1} = A^p \cup \{(i,j)\}$ and $T^{p+1} = T^p \cup \{(i,j)\}$.

This process is completed in n stages.

cost allocation rules ψ

$$\text{mcsts } (N_0, C) \mapsto \psi(N_0, C) \in \mathbf{R}^N$$

B rule (Bird 1976)
Stage p in the Prim algorithm → new agent i_p with new arc (i_p^0, i_p)

$$B_{i_p}(N_0, C) = c_{i_p^0 i_p}$$

(i.e., $B_i(N_0, C) = c_{i^0 i}$, where i^0 is the first node in the unique path in mt T from i to 0)
If there are more than one mts, take the average.
B rule provides a cost allocation in the core of the game (N, v_C)

DK rule (Dutta and Kar 2004)
Based on the Prim algorithm
admit interchange of costs

obligation rule (Kruskal sharing rule) (Tijs et al. 2006, Bergantiños and Kar 2008, Lorenzo and Lorenzo-Freier 2009)

obligation function (sharing function) $o : 2^{N_0} \setminus \{\emptyset\} \to \Delta(N)$

$$\Delta(N) = \{x \in \mathbf{R}^N_+ \mid \sum_{i \in N} x_i = 1\}$$

$$\Delta(S) = \{x \in \mathbf{R}^N_+ \mid \sum_{i \in S} x_i = 1\}$$

- If $0 \in S$, for each $i \in S \setminus \{0\}$, $o_i(S) = 0$.
- If $0 \notin S$, $o(S) \in \Delta(S)$ and for each $R \supset S$ and each $i \in S$, $o_i(R) \leq o_i(S)$.

network $T \to P(T) = \{S_k(T)\}_{k=1}^{n(T)}$ unique partition of N_0 in connected components induced by T

- If $i, j \in S_k(T)$, i and j are connected in T.
- If $i \in S_k(T)$, $j \in S_l(T)$ and $k \neq l$, i and j are not connected in T.

$S(P(T), i)$: element of $P(T)$ to which i belongs obligation rule (Kruskal sharing rule) ϕ^o with obligation function o

$$\phi_i^o(N_0, C) = \sum_{p=1}^{n} c_{i_p j_p}(o_i(S(P(T^{p-1}), i)) - o_i(S(P(T^p), i)))$$

where T^p $(p = 0, \ldots, n)$ is a tree with new arc (i_p, j_p) obtained in Stage p of Kruskal algorithm

ERO rule (Feltkamp et al. 1994, Norde et al. 2004, Tijs et al. 2006)
equal remaining obligation function o^*

$$o^*(S) = \frac{1}{|S|}e^S = \begin{cases} 1/|S| & i \in S \\ 0 & i \notin S \end{cases}, \quad S \subseteq N$$

$$ERO(N_0, C) = \phi^{o^*}(N_0, C)$$

K rule (Kar 2002)
cooperative game (N, v_C) associated with mcsts (N_0, C)

$$v_C(S) = m(S_0, C), \quad S \subseteq N$$

$$K(N_0, C) = Sh(N, v_C) \ \text{ Shapley value}$$

φ rule (Bergantiños and Vidal-Puga 2007)
game (N, v_{C^*}) associated with irreducible form of mcsts (N_0, C)

$$\varphi(N_0, C) = Sh(N, v_{C^*}) = K(N_0, C^*)$$

$$ERO(N_0, C) = \varphi(N_0, C) = K(N_0, C^*) = B(N_0, C^*)$$

existing research

relationships among rules

properties of rules

- core selection (CS)
- strong cost moonotonicity (SCM)
- population monotonicity (PM)
- symmetry (SYM)
- restricted additivity (RA)

research in this talk

mcsts with some structures

- communication structure
- **group structure** (Bergantiños and Gómez-Rúa)
- **layer structure**

→ structures can be reflected in the cost matrix C

2 Minimum cost spanning tree situation with a group structure

mcsts with a group structure (mcsts-g) (N_0, C, G) (Bergantiños and Gómez-Rúa 2008)

- (N_0, C) : mcsts
- $G = \{G^1, G^2, \dots, G^m\}$: partition of N, $G^0 = \{0\}$
- G^k : groups, $k = 0, 1, \dots, m$
- cost of each inner (intragroup) arc is less than or equal to that of any outer (intergroup) arc

$$\max_{i,j \in G^k} c_{ij} \leq \min_{i \in G^k,\ j \notin G^k} c_{ij}$$

intergroup situation (G_0, C^G)

- $G_0 = \{0, 1, \ldots, m\}$
- C^G : $(m+1) \times (m+1)$ cost matrix, $c^G_{kk'} = \min\limits_{i \in G^k, j \in G^{k'}} c_{ij}$

cost allocation for G^k is denoted by $g_k(G_0, C^G)$

intragroup situation (G^k, C^k), $k = 1, \ldots, m$

- $G^k_0 = G^k \cup \{0\}$
- C^k : $(|G^k| + 1) \times (|G^k| + 1)$ cost matrix

$$c^k_{ij} = \begin{cases} c_{ij} & \text{if } 0 \notin \{i, j\} \\ g_k(G_0, C^G) & \text{if } 0 \in \{i, j\} \end{cases} \quad i, j \in G^k$$

cost allocation rule in mcsts-g $\psi : (N_0, C, G) \mapsto \psi(N_0, C, G) \in \mathbf{R}^N$

$$\sum_{i \in N} \psi_i(N_0, C, G) = m(N_0, C)$$

two-phase approach

Phase 1 : intergroup mcsts (G_0, C^G)

cost allocation rule $g : (G_0, C\ \ g) \mapsto \mathbf{R}^G$

$$\sum_{k=1}^{m} g_k(G_0, C^G) = m(G_0, C^G)$$

Condition (A) : $g_k(G_0, C^G) \geq \min_{i,j \in G^k} c_{ij}$, for all $k = 1, \ldots, m$

Phase 2 : intragroup mcsts (G_0^k, C^k) $(k = 1, \ldots, m)$

cost allocation rule $f^k : (G_0^k, C^k) \mapsto \mathbf{R}^{G^k}$

$$\sum_{i \in G^k} f_i^k(G_0^k, C^k) = m(G_0^k, C^k)$$

final allocation rule

$$\psi_i(N_0, C, G) = f_i^k(G_0^k, C^k), \;\; i \in G^k, \;\; k = 1, \ldots, m$$

Lemma Given an mcsts-g (N_0, C, G) we can find an mt T in (N_0, C) satisfying

1. For each $k = 1, \dots, m$, T_{G_k} is an mt in (G^k, C).
2. $T \setminus (\bigcup_{k=1}^{m} T_{G^k})$ defined as $\{(k, k') : \exists i \in G^k, j \in G^{k'} \text{ with } (i, j) \in T\}$ is an mt in (G_0, C^G).
3. Under condition (A), for each $k = 1, \dots, m$ and each $i \in G^k$, $T_{G^k} \cup \{(0, i)\}$ is an mt in (G_0^k, C^k).

Condition (A) : $g_k(G_0, C^G) \geq \min_{i,j \in G^k} c_{ij}$, for all $k = 1, \dots, m$

Corollary Given an mcsts-g (N_0, C, G), under condition (A)

$$\begin{aligned} m(N_0, C) &= m(G_0, C^G) + \sum_{k=1}^{m} m(G^k, C) \\ m(G_0^k, C^k) &= g_k(G_0, C^G) + m(G^k, C) \end{aligned}$$

$$\begin{aligned}\sum_{i\in N}\psi_i(N_0,C,G) &= \sum_{k=1}^{m} m(G_0^k,C^k)=\sum_{k=1}^{m}(g_k(G_0,C^G)+m(G^k,C))\\ &= m(G_0,C^G)+\sum_{k=1}^{m} m(G^k,C)=m(N_0,C)\end{aligned}$$

- two-phase approach provides an allocation rule under condition (A)
- We use the same rule (e.g. ψ) both in intergroup mcsts and intragroup mcsts's → allocation rule ψ in mcsts-g (N_0, C, G)
- In the first phase, B rule, DK rule and obligation rules (including ERO rule) satisfy condition (A), but K rule (Shapley value) does not generally.
- Generally

$$\psi_i(N_0,C,G)\neq\psi_i(N_0,C)$$

- If $G=\{\{i\}\}_{i\in N}$, then $\psi_i(N_0,C,G)=\psi_i(N_0,C)$.
- Properties of several rules can be investigated comparatively. → will be presented on another opportunity

ERO rule φ in both phases
Bergantiños and Gómez-Rúa 2008
$\Pi(N)$: set of permutation (order) π on N ($\pi(i)$ is the position of agent $i \in N$ in the order π)

$$Pre(i,\pi) = \{j \in N \mid \pi(j) < \pi(i)\}$$

permutation $\pi \in \Pi(N)$ is admissible w.r.t. group structure G

$$i, i' \in G^k,\ j \in N,\ \pi(i) < \pi(j) < \pi(i') \ \Rightarrow \ j \in G^k$$

$\Pi^G(N) = \{\pi \in \Pi(N) \mid \pi \text{ is admissible w.r.t. } G\ \}$
(N_0, C^*) : irreducible form of (N_0, C)

Theorem

$$\begin{aligned}
\varphi_i(N_0, C, G) &= Ow_i(N, v_{C^*}, G) \\
&= \frac{1}{|\Pi^G(N)|} \sum_{\pi \in \Pi^G(N)} \left[v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) \right]
\end{aligned}$$

Owen value is a random order value with the weight

$$w_\pi = \begin{cases} \dfrac{1}{|\Pi^G(N)|} & \text{if } \pi \in \Pi^G(N) \\ 0 & \text{otherwise} \end{cases}$$

Bergantiños and Kar 2008

random order value over (N, v_{C^*}) is realized by an obligation rule ϕ^o for (N_0, C)

$\varphi(N_0, C, G) = \phi^o(N_0, C)$ for some obligation function o for N

→ we would like to obtain this o

Lorenzo and Lorenzo-Freier 2009

Theorem If an allocation rule $\psi(N_0, C)$ is realized as an obligation rule $\phi^{\hat{o}}(N_0, C)$, i.e.,

$$\psi(N_0, C) = \phi^{\hat{o}}(N_0, C),$$

then

$$\hat{o}(S) = \psi(S_0, \hat{C}), \quad S \subseteq N, S \neq \emptyset$$

with the cost matrix $\hat{C} = (\hat{c}_{ij})$

$$\hat{c}_{ij} = \begin{cases} 1 & \text{if } i \text{ or } j = 0 \\ 0 & \text{otherwise} \end{cases}, \text{ i.e. } C = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

(N_0, C, G) : mcsts-g, $S \subseteq N$
$G(S) = \{G^1 \cap S, \ldots, G^m \cap S\}$ (deleted if empty) : group structure induced by S
intergroup mcsts $((G(S))_0, \hat{C}^{G(S)})$

$$\hat{c}_{kl}^{G(S)} = \begin{cases} 1 & \text{if } k \text{ or } l = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_k((G(S))_0, C^{G(S)}) = \frac{1}{|G(S)|}, \quad \forall k$$

intragroup mcsts $(S_0^k, \hat{C}^k)$ for $S^k = S \cap G^k \neq \emptyset$

$$\varphi_i(S_0^k, \hat{C}^k) = \frac{1}{|S^k||G(S)|}, \ \forall i \in S^k$$

Theorem Let (N_0, C, G) be an mcsts-g. Then

$$\varphi(N_0, C, G) = \phi^{\hat{o}}(N_0, C)$$

$$\hat{o}_i(S) = \begin{cases} \dfrac{1}{|S \cap G^k||G(S)|} & \text{if } i \in S \cap G^k \\ 0 & \text{otherwise} \end{cases}$$

two-phase ERO rule in mcsts-g (N_0, C, G) is realized by an ordinary obligation rule in mcsts (N_0, C)

3 Minimum cost spanning tree situation with a layer structure

mcsts with a layer structure (mcsts-l) (N_0, C, L)

- (N_0, C) : mcsts
- $L = \{L^1, L^2, \ldots, L^m\}$: partition of N, $L^0 = \{0\}$
- L^k : layers, $k = 0, 1, \ldots, m$
- cost conditions

1. cost of each arc in the same layer is higher than that of an arc between different layers

$$k, k' \in \{0, \ldots, m\}, k \neq k', i, j \in L^k, i' \in L^{k'} \Rightarrow c_{ij} > c_{ii'}$$

2. cost of each arc between adjacent layers is lower than that between farther layers

$$k, k' \in \{0, \ldots, m\}, k' < k-1, i \in L^k, j \in L^{k-1}, j' \in L^{k'} \Rightarrow c_{ij} < c_{ij'}$$
$$k, k' \in \{0, \ldots, m\}, k' > k+1, i \in L^k, j \in L^{k+1}, j' \in L^{k'} \Rightarrow c_{ij} < c_{ij'}$$

3. cost in the upper layer is relatively higher

$$k \in \{1, \ldots, m-1\}, i, j \in L^k, i' \in L^{k-1}, j' \in L^{k+1} \Rightarrow \max\{c_{i'i}, c_{i'j}\} < \max\{c_{ij'}, c_{jj'}$$

Characterization of mt in mcsts-l

Theorem Let (N_0, C, L) be an mcsts-l. For any mt $T = \{(i^0, i) \mid i \in N\}$ of (N_0, C), if $i \in L^k$, then $i^0 \in L^{k-1}$ (i.e., mt is a hierarchical tree having 0 as the root).

strict layer structure

1. cost of each arc in the same layer is higher than that of an arc between different layers

$$k, k' \in \{0, \ldots, m\}, k \neq k', i, j \in L^k, i' \in L^{k'} \Rightarrow c_{ij} > c_{ii'}$$

2. cost of each arc between farther layers is higher

$$\begin{gathered} k, k', k'' \in \{0, \ldots, m\}, k'' < k' < k, i \in L^k, i' \in L^{k'}, i'' \in L^{k''} \Rightarrow c_{ii'} < c_{ii''} \\ k, k', k'' \in \{0, \ldots, m\}, k < k' < k'', i \in L^k, i' \in L^{k'}, i'' \in L^{k''} \Rightarrow c_{ii'} < c_{ii''} \end{gathered}$$

3. cost in the upper layer is relatively higher

$$\begin{gathered} k, k', k'' \in \{1, \ldots, m-1\}, k' < k < k'', i, j \in L^k, i' \in L^{k'}, j' \in L^{k''} \\ \Rightarrow \max\{c_{i'i}, c_{i'j}\} < \max\{c_{ij'}, c_{jj'}\} \end{gathered}$$

(N_0, C, L) : mcsts-l, $S \subset N$
layer structure induced by S

$L(S) = \{L^1 \cap S, \ldots, L^m \cap S\} = \{L^1(S), \ldots, L^{m'}(S)\}$ (deleted if empty and renumbered

Theorem Let (N_0, C, L) be an mcsts-l with a strict layer structure L and $S \subseteq N$. Then the layer structure $L(S)$ induced by S is also a (strict) layer structure.

Corollary Let (N_0, C, L) be an mcsts-l with a strict layer structure L and $S \subseteq N$. For any mt $T = \{(i^0, i) \mid i \in S\}$ of (S_0, C), if $i \in L^k(S)$, then $i^0 \in L^{k-1}(S)$ (i.e., mt is a hierarchical tree having 0 as the root).

obligation function o associated with a layer structure L

$$k(S) := \min\{k \mid i \in L^k \cap S\},\ S \subseteq N$$
$$\underline{S} := \{i \in S \mid i \in L^{k(S)}\}$$
$$o_i^L(S) = \begin{cases} \dfrac{1}{|\underline{S}|} & \text{if } i \in L^{k(S)} \\ 0 & \text{otherwise} \end{cases}$$

Theorem Let (N_0, C, L) be an mcsts-l and o^L be an obligation function for N associated with L. Then

$$\phi^{o^L}(N_0, C) = B(N_0, C).$$

monotone layer structure $L = \{L^1, \ldots, L^m\}$ (monotone $\Rightarrow$ strict)

For any sequence of agents $\{i_h\}_{h=1}^{l}$ satisfying $i_h \in L^h$ for $h = 1, \ldots, l$, we have

$$c_{0i_1} \leq c_{i_1 i_2} \leq \ldots \leq c_{i_{l-1} i_l}.$$

Theorem Let (N_0, C, L) be an mcsts-l with a monotone layer structure L and o be an arbitrary obligation function for N. Then

$$\phi^o(N_0, C) = B(N_0, C) = DK(N_0, C) = B(N_0, C^*) = K(N_0, C^*).$$

Moreover, in the mcsts-l (N_0, C^*, L), we have the mt $\{(i_{k-1}, i_k) | k = 1, \ldots, m\}$ with $c_{i_{k-1} i_k} = \min_{j_{k-1} \in L^{k-1}, j_k \in L^k} c_{j_{k-1} j_k}$ (trunk + intralayer arcs).

5. Conclusion

minimum cost spanning tree situations with some structures

- group structure

  two-phase approach can be justified under condition (A)
  two-phase ERO rule can be realized by an ordinary obligation rule

- layer structure

  hierarchical minimum cost spanning tree
  relationships between Bird rule and obligation rules

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