On a cooperative game in partition function form motivated by a cost allocation problem

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Introduction

- **Cost allocation problem** Several actors try to obtain a bundle of goods, implementing respective development projects. They can create coalitions to decrease costs of the projects. Problem: how to allocate the cost among the actors and among the goods.

- **References:** Littlechild, Thompson; Young, Okada, Hashimoto; Tanino; Seo, Sakawa; Legros; Krus and Bronisz, Fernández, Hinojosa, Puerto; Cruijssen, Cools, Dullaert; Matsubayashi, Umezawa, Masuda, Nishino; Krajewska, Kopfer; Cruijssen, Cools, Dullaert;

- **A cooperative game in characteristic function form (CFF game) approach** - it was assumed that the payoff of each coalition depends only on the players who create it.
Introduction

This paper: application of a cooperative game in partition function form (PFF game); assumption: the gain of any coalition depends also on the coalition structure the other players.

Direction of research:

- formulation of the PFF game motivated by the cost allocation problem,
- formulation and analysis of solution concepts,
- looking for a CFF game related to the PFF game, such that solutions of the games coincide,
- utilization of solution concepts like cores and nucleoli derived for the CFF game in decision support - using Kruś, Bronisz (2000) ideas.

References to the PFF games

Thrall and Lucas; Chander, Tulkens; Huang, Sjöström; Auman, Peleg; Kóczy.
Notation and initial assumptions

\( \mathcal{N} = \{1, \ldots, n\} \) - a finite set of (actors) players,
\( \mathcal{N} \) - the set of all nonempty subsets of \( \mathcal{N} \),
\( S \) - a given coalition of players, \( S \in \mathcal{N} \).

Each player is interested in a bundle of goods \( M = \{1, \ldots, m\} \),
which can be obtained by covering some costs.

For any \( S \in \mathcal{N} \) let \( P_S = \{P_1, \ldots, P_r\} \) be a partition of \( S \), i.e.

\[
\bigcup_{i=1}^{r} P_i = S, \quad \forall j \ P_j \neq 0, \quad \forall k \ P_j \cap P_k = \emptyset \text{ if } k \neq j, \quad (1)
\]

and let \( \Pi_S \) denote the set of all partitions of \( S \), \( \Pi \) denote \( \Pi_N \) and
\( P_I = \{\{1\}, \{2\}, \ldots, \{n\}\} \).
Let \( z_i = (z_{i1}, \ldots, z_{im}) \in \mathbb{R}_+^m \) denote a bundle of goods desired by a player \( i \) (\( z_{ij} \) is the desired volume of the good \( j \)) for \( j \in M, i \in N \), and \( z = (z_1, \ldots, z_n) \in \mathbb{R}^{NM} = \mathbb{R}_+^{n \times m} \), where: 
\( \mathbb{R}^{MN} \) is the space of goods desired by the grand coalition \( N \),

For each coalition \( P_k \subseteq N \) and partition \( P \in \Pi \), such that \( P_k \in P \) there are given functions \( f_{P_k,P}(z_{P_k}) \) describing the cost required to obtain a bundle \( z_{P_k} = (z_{P_k,1}, \ldots, z_{P_k,m}) \) of \( m \) goods desired by the coalition. We assume, that the functions depend on the total amount \( z_{P_k} \) of goods and do not depend on a division of them among players in \( P_k \), but depend on the partition of the other players.
Let a bundle of goods \( z = (z_1, \ldots, z_n) \in \mathbb{R}^{NM} \) desired by all players \( i \in N \) and cost functions \( f_{P_k}(z_{P_k}) \) for all coalitions \( P_k \) and partitions \( P \in \Pi \) are given.

**Definition of the PFF game**

A cooperative game in partition function form (PFF game) is defined by a pair \((N, F)\), where \( N \) is the set of players and \( F \) is a function which assigns \( r \)-dimensional real vector

\[
F_P = (F_{P}(P_1), ..., F_{P}(P_r))
\]

to each partition \( P \in \Pi \),

\( P = \{P_1, P_2, ..., P_r\} \), where

\[
F_{P}(P_k) = \sum_{i \in P_k} f_{\{i\}, P_l}(z_i) - f_{P_k, P}(\sum_{i \in P_k} z_i) \quad \text{for } k = 1, ..., r.
\]

\( F_{P}(P_k) \) describes the benefit the players can obtain acting together in a coalition \( P_k \) in comparison to their individual actions.
Formulation of the PFF game

For each $S \subseteq N$ and each partition $P_S \in \Pi_S$:

$$v(S) = \min_{\{P \in \Pi : S \in P\}} F_P(S), \quad v(\emptyset) = 0 \quad (2)$$

denotes the guaranteed worth of a coalition $S$ independent on the behavior of the players,

$$u(P_S) = \min_{\{P \in \Pi : P_S \subseteq P\}} \sum_{T \in P_S} F_P(T) \quad (3)$$

- the amount which is guaranteed for the players arranged in $P_S$ independent on the behavior of the others players,

$$\bar{v}(S) = \max_{\{P_S \in \Pi_S\}} u(P_S), \quad \bar{v}(\emptyset) = 0 \quad (4)$$

- the maximal amount which is guaranteed for the players in $S$ independent on the behavior of the other players.
The following inequalities hold for any PFF game \((N, F)\):

\[
\overline{v}(\{i\}) = v(\{i\}) \quad \text{for each } i \in N, \tag{5}
\]

\[
v(S) \leq \overline{v}(S) \quad \text{for each } S \subset N, \tag{6}
\]

\[
u(P_S) + u(P_T) \leq u(P_S \cup P_T)
\]

for each \(P_S \in \Pi_S, \ P_T \in \Pi_T\) \hspace{1cm} \tag{7}

where \(S, T \subset N\) such that \(S \cap T = \emptyset\).

\[
\overline{v}(S) + \overline{v}(T) \leq \overline{v}(S \cup T)
\]

for each \(S, T \subset N\) such that \(S \cap T = \emptyset\) \hspace{1cm} \tag{8}
A vector $x = (x_1, \ldots, x_n)$ is called an **imputation** if

\[ x_i \geq \overline{v}(\{i\}) \quad \text{for each} \quad i \in N, \quad \text{(9)} \]

\[ \sum_{i \in N} x_i = \sum_{S \in P} F_P(S) \quad \text{for some} \quad P \in \Pi. \quad \text{(10)} \]
Definition

Let $S$ be a nonempty subset of $N$ and let $x, y \in R$. We say that $x$ dominates $y$ via $S$ (denoted: $x \text{ Dom}_S y$) if

$$x_i > y_i \quad \text{for each } i \in S,$$

(11)

and there exists $P_S \in \Pi_S$ such that

$$\sum_{i \in S} x_i \leq u(P_S),$$

(12)

for some $P \in \Pi$ such that $P_S \subset P$. 

$$\sum_{i \in N} x_i = \sum_{T \in P} F_P(T)$$

(13)
We say that $x$ dominates $y$ (denoted by $x \ Dom y$) if $x \ Dom_S y$ for some $S \subset N$. For $X \subset R$

$$\text{Dom}_S X = \{ y \in R : x \ Dom_S y \text{ for some } x \in X \},$$

$$\text{Dom} X = \{ y \in R : x \ Dom y \text{ for some } x \in X \}.$$

**Definition**

A set of imputations $K$ is a **stable set** if

$$K \cap \text{Dom} K = \emptyset,$$  \hspace{1cm} (14)

$$K \cup \text{Dom} K = R.$$ \hspace{1cm} (15)

**Definition**

A set of imputations $C$ is the **core** if

$$C = R \setminus \text{Dom} R.$$ \hspace{1cm} (16)
For each partition $P \in \Pi$ in a game $(N, F)$ let

$$\|P\| = \sum_{S \in P} F_P(S).$$  \hspace{1cm} (17)

**Definition**

Any imputation $x$ has the property of **group rationality** if

$$\sum_{i \in N} x_i = \max_{\{P \in \Pi\}} \|P\|.$$  \hspace{1cm} (18)
**Theorem 1**

Let $C$ denote the core of a game $(N, F)$. Each imputation $x \in C$ has the property of group rationality.

**Theorem 2**

The core of the game $(N, F)$ is equal to the core of the cooperative game in characteristic function form $(N, \overline{v})$, i.e. it satisfies the following conditions:

\[
\sum_{i \in S} x_i \geq \overline{v}(S) \quad \text{for each } S \subset N, \quad (19)
\]

\[
\sum_{i \in N} x_i = \overline{v}(N). \quad (20)
\]

Compare equation (4), conditions $\overline{v}(\emptyset) = 0$ and (8).
Theorem 3

For an $n$-person game with $\tilde{P} = \{N\}$ such that $\|\tilde{P}\| > \|P\|$ for each $P \in \Pi$, $P \neq \tilde{P}$, there exists unique stable set $K = R^{\{N\}} = R^{\text{max}}$. Each imputation $x \in K$ has the property of group rationality.

Theorem 4

If $\hat{P} = \{\hat{P}_1, \ldots, \hat{P}_r\}$ is a partition of $N$ such that $\|\hat{P}\| > \|P\|$ for each $P \in \Pi$, $P \neq \hat{P}$ then the game $(N, F)$ has the same stable sets as a game in characteristic function form $(N, \hat{v})$ defined by

$$
\hat{v}(S) = \overline{v}(T) + \sum_{i \in S \setminus T} \overline{v}({i})
$$

for each $S \subset N$, $\hat{v}(\emptyset) = 0,

(21)

where: $T \subset S$, $T = \bigcup_{i=1}^{r} \{\hat{P}_i : \hat{P}_i \subset S\}$. 

The cooperative PFF game model for a cost allocation problem has been formulated for a given bundle of goods desired by players. Proposed solutions are considered as parametric with respect to volumes of the goods.

In the PFF game the payoff of any coalition depends on the coalition structure of the other payers.

The core and the stable set of the PFF game game \((N, F)\) can be derived as the core and the stable set of the respective CFF games (Theorem 2 and 4).

The core can be derived according to the conditions (16) and it can be presented to the players as the set defining frames of their negotiations.

Considering the respective CFF game, different nucleoli concepts can be derived and proposed for discussion, according to the ideas by Krus, Bronisz (2000).