On a cooperative game in partition function form motivated by a cost allocation problem

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Outline

Introduction

- **Cost allocation problem** Several actors try to obtain a bundle of goods, implementing respective development projects. They can create coalitions to decrease costs of the projects. Problem: how to allocate the cost among the actors and among the goods.
- **References:** Littlechild, Thompson; Young, Okada, Hashimoto; Tanino; Seo, Sakawa; Legros; Krus and Bronisz, Fernández, Hinojosa, Puerto; Cruijssen, Cools, Dullaert; Matsubayashi , Umezawa, Masuda, Nishino; Krajewska, Kopfer; Cruijssen, Cools, Dullaert;
- **A cooperatve game in characteristic function form (CFF game) approach** - it was assumed that the payoff of each coalition depends only on the players who create it.

Introduction

This paper: application of a cooperative game in partition function form (PFF game); assumption: the gain of any coalition depends also on the coalition structure the other players.

Direction of research:

- formulation of the PFF game motivated by the cost allocation problem,
- formulation and analysis of solution concepts,
- looking for a CFF game related to the PFF game, such that solutions of the games coincide,
- utilization of solution concepts like cores and nucleoli derived for the CFF game in decision support - using Kruś, Bronisz (2000) ideas.

References to the PFF games

Thrall and Lucas; Chander, Tulkens; Huang, Sjöström; Auman, Peleg; Kóczy.

Notation and initial assumptions

$N = \{1, \ldots, n\}$ - a finite set of (actors) players,
$\mathcal{N}$ - the set of all nonempty subsets of N,
S - a given coalition of players, $S \in \mathcal{N}$.
Each player is interested in a bundle of goods $M = \{1, \ldots, m\}$, which can be obtained by covering some costs.
For any $S \in \mathcal{N}$ let $P_S = \{P_1, ..., P_r\}$ be a partition of S, i.e.

$$\bigcup_{i=1}^{r} P_i = S, \; \forall j \; P_j \neq 0, \; \forall k \; P_j \cap P_k = \emptyset \text{ if } k \neq j, \tag{1}$$

and let Π_S denote the set of all partitions of S, Π denote Π_N and $P_I = \{\{1\}, \{2\}, \ldots, \{n\}\}$.

Notation and initial assumptions

Let $z_i = (z_{i1}, \ldots, z_{im}) \in \boldsymbol{R}_+^m$ denote a bundle of goods desired by a player i (z_{ij} is the desired volume of the good j) for $j \in M$, $i \in N$, and $\mathbf{z} = (z_1, \ldots, z_n) \in \boldsymbol{R}^{NM} = \boldsymbol{R}_+^{n \times m}$, where:
$\boldsymbol{R}^{MN}$ is the space of goods desired by the grand coalition N,

For each coalition $P_k \subseteq N$ and partition $P \in \Pi$, such that $P_k \in P$ there are given functions $f_{P_k,P}(z_{P_k})$ describing the cost required to obtain a bundle $z_{P_k} = (z_{P_k,1}, ..., z_{P_k,m})$ of m goods desired by the coalition. We assume, that the functions depend on the total amount z_{P_k} of goods and do not depend on a division of them among players in P_k, but depend on the partition of the other players.

Let a bundle of goods $\mathbf{z} = (z_1, \ldots, z_n) \in \boldsymbol{R}^{NM}$ desired by all players $i \in N$ and cost functions $f_{P_k,P}(z_{P_k})$ for all coalitions P_k and partitions $P \in \Pi$ are given.

Definition of the PFF game

A cooperative game in partition function form (PFF game) is defined by a pair (N, F), where N is the set of players and F is a function which assigns r-dimensional real vector
$F_P = (F_P(P_1), \ldots, F_P(P_r))$ to each partition $P \in \Pi$,
$P = \{P_1, P_2, \ldots, P_r\}$, where
$F_P(P_k) = \sum_{i \in P_k} f_{\{i\},P_I}(z_i) - f_{P_k,P}(\sum_{i \in P_k} z_i)$ for $k = 1, \ldots, r$.

□

$F_P(P_k)$ describes the benefit the players can obtain acting together in a coalition P_k in comparison to their individual actions.

For each $S \subseteq N$ and each partition $P_S \in \Pi_S$:

$$v(S) = \min_{\{P \in \Pi : S \in P\}} F_P(S), \quad v(\emptyset) = 0 \tag{2}$$

denotes the guaranteed worth of a coalition S independent on the behavior of the players,

$$u(P_S) = \min_{\{P \in \Pi : P_S \subset P\}} \sum_{T \in P_S} F_P(T) \tag{3}$$

- the amount which is guaranteed for the players arranged in P_S independent on the behavior of the others players,

$$\overline{v}(S) = \max_{\{P_S \in \Pi_S\}} u(P_S), \quad \overline{v}(\emptyset) = 0 \tag{4}$$

- the maximal amount which is guaranteed for the players in S independent on the behavior of the other players.

The following inequalities hold for any PFF game (N, F) :

$$\overline{v}(\{i\}) = v(\{i\}) \quad \text{for each } i \in N, \tag{5}$$

$$v(S) \leq \overline{v}(S) \quad \text{for each } S \subset N, \tag{6}$$

$$u(P_S) + u(P_T) \leq u(P_S \cup P_T) \quad \text{for each } P_S \in \Pi_S,\ P_T \in \Pi_T \quad \text{where } S, T \subset N \text{ such that } S \cap T = \emptyset. \tag{7}$$

$$\overline{v}(S) + \overline{v}(T) \leq \overline{v}(S \cup T) \quad \text{for each } S, T \subset N \text{ such that } S \cap T = \emptyset \tag{8}$$

Definition

A vector $x = (x_1, \ldots x_n)$ is called an **imputation** if

$$x_i \geq \overline{v}(\{i\}) \quad \text{for each } i \in N, \tag{9}$$

$$\sum_{i \in N} x_i = \sum_{S \in P} F_P(S) \quad \text{for some } P \in \Pi. \tag{10}$$

Definition

Let S be a nonempty subset of N and let $x, y \in R$. We say that x **dominates** y **via** S (denoted: $x \; Dom_S \; y$) if

$$x_i > y_i \quad \text{for each } i \in S, \tag{11}$$

and there exists $P_S \in \Pi_S$ such that

$$\sum_{i \in S} x_i \leq u(P_S), \tag{12}$$

$$\begin{array}{l} \sum_{i \in N} x_i = \sum_{T \in P} F_P(T) \\ \text{for some } P \in \Pi \text{ such that } P_S \subset P. \end{array} \tag{13}$$

We say that x dominates y (denoted by $x\,Dom\,y$) if $x\,Dom_S\,y$ for some $S \subset N$. For $X \subset R$

$$Dom_S\,X = \{y \in R : \; x\,Dom_S\,y \text{ for some } x \in X\},$$
$$Dom\,X = \{y \in R : \; x\,Dom\,y \quad \text{for some } x \in X\}.$$

Definition

A set of imputations K is a **stable set** if

$$K \cap Dom\,K = \emptyset, \tag{14}$$
$$K \cup Dom\,K = R. \tag{15}$$

Definition

A set of imputations C is the **core** if

$$C = R \setminus Dom\,R. \tag{16}$$

For each partition $P \in \Pi$ in a game (N, F) let

$$\|P\| = \sum_{S \in P} F_P(S). \tag{17}$$

Definition

Any imputation x has the property of **group rationality** if

$$\sum_{i \in N} x_i = \max_{\{P \in \Pi\}} \|P\|. \tag{18}$$

Theorem 1

Let C denote the core of a game (N, F). Each imputation $x \in C$ has the property of group rationality.

Theorem 2

The core of the game (N, F) is equal to the core of the cooperative game in characteristic function form $(N, \overline{v})$, i.e. it satisfies the following conditions:

$$\sum_{i \in S} x_i \geq \overline{v}(S) \quad \text{for each } S \subset N, \tag{19}$$

$$\sum_{i \in N} x_i = \overline{v}(N). \tag{20}$$

Compare equation (4), conditions $\overline{v}(\emptyset) = 0$ and (8).

Theorem 3

For an n-person game with $\tilde{P} = \{N\}$ such that $\|\tilde{P}\| > \|P\|$ for each $P \in \Pi$, $P \neq \tilde{P}$, there exists unique stable set $K = R^{\{N\}} = R^{\max}$.
Each imputation $x \in K$ has the property of group rationality.

Theorem 4

If $\hat{P} = \{\hat{P}_1, \dots, \hat{P}_r\}$ is a partition of N such that $\|\hat{P}\| > \|P\|$ for each $P \in \Pi$, $P \neq \hat{P}$ then the game (N, F) has the same stable sets as a game in characteristic function form $(N, \hat{v})$ defined by

$$\begin{gathered} \hat{v}(S) = \overline{v}(T) + \textstyle\sum_{i \in S \setminus T} \overline{v}(\{i\}) \\ \text{for each } S \subset N, \quad \hat{v}(\emptyset) = 0, \end{gathered} \tag{21}$$

$$\text{where: } T \subset S, \qquad T = \bigcup_{i=1}^{r} \{\hat{P}_i : \ \hat{P}_i \subset S\}.$$

Final remarks

- The cooperative PFF game model for a cost allocation problem has been formulated for a given bundle of goods desired by players. Proposed solutions are considered as parametric with respect to volumes of the goods.
- In the PFF game the payoff of any coalition depends on the coalition structure of the other payers.
- The core and the stable set of the PFF game game (N, F) can be derived as the core and the stable set of the respective CFF games (Theorem 2 and 4).
- The core can be derived according to the conditions (16) and it can be presented to the players as the set defining frames of their negotiations.
- Considering the respective CFF game, different nucleoli concepts can be derived and proposed for discussion, according to the ideas by Krus, Bronisz (2000).