Interactive cone contraction method for multiple objective optimization problems

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Interactive cone contraction method for MOO problems

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4. Conclusions and further research
Multiple-Objective Optimization Problem

**Characteristics**

- **Solutions** described by evaluation vectors
- Several objective functions to be optimized simultaneously
- **Conflicting viewpoints**
- **Minimize** \([f_1(x), f_2(x), \ldots, f_k(x)]\)
  
  s.t. \(x \in S\)
- Assumption: **less is preferred to more**
- Find the **best option**

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Pareto optimal solutions

- Decision vector $x \in S$ is called Pareto optimal if and only if there is no other $y \in S$ such that $f_i(y) \leq f_i(x)$, $i = 1, \ldots, k$ and $j = \{1, \ldots, k\}$, $f_j(y) < f_j(x)$

- Equally desirable in the mathematical sense

- Different trade-offs

- Synonyms: Pareto-optimal, non-dominated, efficient, non-inferior

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Pareto optimal solutions

- Pareto optimal set can be generated using some non-interactive a posteriori techniques or different evolutionary (EMO) approaches
- Dominance relation is too poor
- Finding a final solution necessitates participation of the DM
Preference information

Reference point

- **Arbitrary reference point** denoted by $\bar{z}$
- **Desired** objective function values that the DM would like to achieve (aspiration) or should be achieved (reservation)
- Not necessarily equal to ideal or utopian objective vector
- **Natural** way of expressing desires
**Preference information**

**Pairwise comparisons**
- Comparison of non-dominated solutions from a current sample
- Strict ($x^1 \succ x^2$) or weak ($x^1 \succeq x^2$) preference relation
- Holistic judgements, very simple, easy to understand, natural, not too demanding of cognitive effort
- Exercising decisions
Achievement scalarizing function

Characteristics

- Used to **project** a reference point onto the set of efficient solutions

\[ s(x, \lambda, f) = \max_i \{ \lambda_i (f_i(x) - \bar{z}_i) \} + \rho \sum_{i=1}^{k} \lambda_i (f_i(x) - \bar{z}_i) \]

- Similar form to an **augmented weighted Chebyshev norm**

- **The less** the value of the function, **the less** the distance

- Some projection directions may be **more desirable**
General idea of the method

Examplary MOO problem

- Two objectives that are to be minimized
- Reference set consisting of three solutions
- Reference point in the origin of the evaluation space
First iteration

- First comparison: $x^2 \succ x^1$
- Indication of the preferred subspace in the evaluation space
- Solutions $x^2$ and $x^3$ inside, $x^1$ outside
General idea of the method

Second iteration

- Second comparison: $x^2 \succ x^3$
- Indication of the preferred subspace in the evaluation space
- Solution $x^2$ inside, $x^1$ and $x^3$ outside
After two iterations

- First comparison: $x^2 \succ x^1$
- Second comparison: $x^2 \succ x^3$
- Many solutions are not longer seen as potential best choices
General idea of the method

- Direction of the isoquants of all compatible achievement scalarizing functions
- Combinations of weights ensuring that reference solutions are compared in the same way as done by the DM
- Search for the boundary weighting vectors
Interactive cone contraction method

Main Principles

- **Systematic dialogue** with the decision maker
- **Specification of a reference point and pairwise comparisons** of non-dominated solutions from a current sample
- **Incorporation** of preference information into weights in the achievement scalarizing function
- **Systematic reduction** of the feasible region (cone) and of the set of solutions that can be chosen as the final one
Step 1

- **Compute the Pareto optimal set** $P(S)_0$ of multiple objective optimization problem

- **66 solutions** satisfying the following condition:
  $$f_1(x) + f_2(x) + f_3(x) = 0.5$$

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Step 2

- Ask the DM to specify the reference point for the current iteration, $\bar{z}_q$.
- The reference point is situated in the origin of the evaluation space, $[0.0, 0.0, 0.0]$. 

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Step 3

- Ask the DM to provide preference information in form of **pairwise comparisons** for the solutions chosen from $P(S)_q$

- First comparison:
  $$x^{24} = [0.10, 0.10, 0.30] \succ x^{43} = [0.20, 0.20, 0.10]$$

- The transition from the preference relations $\succ$ and $\succeq$ to the function values:
  - $x^1 \succ x^2$ implicates that $s(x^1, \lambda^q, f) < s(x^2, \lambda^q, f)$,
  - $x^1 \succeq x^2$ implicates that $s(x^1, \lambda^q, f) \leq s(x^2, \lambda^q, f)$.

- $s(x^{24}, \lambda^q, f) < s(x^{43}, \lambda^q, f)$
Optimize the boundary combinations of weights $\lambda_i, \ i = 1, \ldots, k$ of the compatible achievement scalarizing function

$$\min M\lambda_{r,\text{seq}}^q + N\lambda_{r+1,\text{seq}}^q + \cdots + \lambda_{r+k-1,\text{seq}}^q$$

[a] $s(x^1, \lambda_{\text{seq}}^q, f) - s(x^2, \lambda_{\text{seq}}^q, f) < 0 \Leftrightarrow x^1, x^2 \in P(S)_q : x^1 \succ x^2$

[b] $s(x^l, \lambda_{\text{seq}}^q, f) = \alpha(x^l) + \rho \sum_{i=1}^{k} \lambda_{i,\text{seq}}^q (f_i(x^l) - \bar{z}_{q,i}) : l = 1, 2$

[c] $\alpha(x^l) \geq \lambda_{i,\text{seq}}^q (f_i(x^l) - \bar{z}_{q,i}) : l = 1, 2, \ i = 1, \ldots, k$

[d] $\sum_{i=1}^{k} \lambda_{i,\text{seq}}^q = 1$

[e] $\lambda_{j,\text{seq}}^q \geq \lambda_{j,\text{seq}}^{q-1} : j = r, \ldots, r + k - 1$

[f] $\lambda_{r+k,\text{seq}}^q \leq \lambda_{r+k,\text{seq}}^{q-1}$

where $M \gg N \gg \ldots \gg 1$
Step 4

Optimize the boundary combinations of weights $\lambda_i$, $i = 1, \ldots, k$ of the compatible achievement scalarizing function

$$
\min M\lambda_{r,\text{seq}}^q + N\lambda_{r+1,\text{seq}}^q + \cdots + \lambda_{r+k-1,\text{seq}}^q
$$

\begin{align*}
[a] & \quad s(x^1, \lambda_{\text{seq}}^q, f) - s(x^2, \lambda_{\text{seq}}^q, f) < 0 \iff x^1, x^2 \in P(S)_q : x^1 \succ x^2 \\
[b] & \quad s(x^l, \lambda_{\text{seq}}^q, f) = \alpha(x^l) + \rho \sum_{i=1}^k \lambda_{i,\text{seq}}^q (f_i(x^l) - \bar{z}_{q,i}) : l = 1, 2 \\
[c] & \quad \alpha(x^l) \geq \lambda_{i,\text{seq}}^q (f_i(x^l) - \bar{z}_{q,i}) : l = 1, 2, i = 1, \ldots, k \\
[d] & \quad \sum_{i=1}^k \lambda_{i,\text{seq}}^q = 1 \\
[e] & \quad \lambda_{j,\text{seq}}^q \geq \lambda_{j,\text{seq}}^{q-1} : j = r, \ldots, r + k - 1 \\
[f] & \quad \lambda_{r+k,\text{seq}}^q \leq \lambda_{r+k,\text{seq}}^{q-1}
\end{align*}

where $M >> N >> \ldots >> 1$
Step 4

Optimize the boundary combinations of weights $\lambda_i$, $i = 1, \ldots, k$ of the compatible achievement scalarizing function

$$
\min M\lambda_{r,\text{seq}}^q + N\lambda_{r+1,\text{seq}}^q + \cdots + \lambda_{r+k-1,\text{seq}}^q
$$

[a] $s(x^1, \lambda_{\text{seq}}^q, f) - s(x^2, \lambda_{\text{seq}}^q, f) < 0 \iff x^1, x^2 \in P(S)_q : x^1 \succ x^2$

[b] $s(x^l, \lambda_{\text{seq}}^q, f) = \alpha(x^l) + \rho \sum_{i=1}^k \lambda_{i,\text{seq}}^q (f_i(x^l) - \bar{z}_{q,i}) : l = 1, 2$

[c] $\alpha(x^l) \geq \lambda_{i,\text{seq}}^q (f_i(x^l) - \bar{z}_{q,i}) : l = 1, 2, i = 1, \ldots, k$

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[e] $\lambda_{j,\text{seq}}^q \geq \lambda_{j,\text{seq}}^{q-1} : j = r, \ldots, r + k - 1$

[f] $\lambda_{r+k,\text{seq}}^q \leq \lambda_{r+k,\text{seq}}^{q-1}$

where $M >> N >> \ldots >> 1$
Optimize the boundary combinations of weights $\lambda_i$, $i = 1, \ldots, k$ of the compatible achievement scalarizing function

$$
\min M\lambda_q^{r, \text{seq}} + N\lambda_q^{r+1, \text{seq}} + \cdots + \lambda_q^{r+k-1, \text{seq}}
$$

\[ a \] $s(x^1, \lambda_q^{\text{seq}, f}) - s(x^2, \lambda_q^{\text{seq}, f}) < 0 \iff x^1, x^2 \in P(S)_q : x^1 \succ x^2$

\[ b \] $s(x^l, \lambda_q^{\text{seq}, f}) = \alpha(x^l) + \rho \sum_{i=1}^{k} \lambda_q^{i, \text{seq}}(f_i(x^l) - \bar{z}_{q,i}) : l = 1, 2$

\[ c \] $\alpha(x^l) \geq \lambda_q^{i, \text{seq}}(f_i(x^l) - \bar{z}_{q,i} : l = 1, 2, i = 1, \ldots, k$

\[ d \] $\sum_{i=1}^{k} \lambda_q^{i, \text{seq}} = 1$

\[ e \] $\lambda_q^{j, \text{seq}} \geq \lambda_q^{j-1, \text{seq}} : j = r, \ldots, r + k - 1$

\[ f \] $\lambda_q^{r+k, \text{seq}} \leq \lambda_q^{r+k-1, \text{seq}}$

where $M >> N >> \ldots >> 1$
Step 4

Optimize the boundary combinations of weights \( \lambda_i, \, i = 1, \ldots, k \) of the compatible achievement scalarizing function

\[
\min M\lambda_{r, \text{seq}}^q + N\lambda_{r+1, \text{seq}}^q + \ldots + \lambda_{r+k-1, \text{seq}}^q
\]

\[a\] \( s(x^1, \lambda_{\text{seq}}^q, f) - s(x^2, \lambda_{\text{seq}}^q, f) < 0 \iff x^1, x^2 \in P(S)_q : x^1 \succ x^2 \)

\[b\] \( s(x^l, \lambda_{\text{seq}}^q, f) = \alpha(x^l) + \rho \sum_{i=1}^{k} \lambda_{i, \text{seq}}^q (f_i(x^l) - \bar{z}_{q,i}) : l = 1, 2 \)

\[c\] \( \alpha(x^l) \geq \lambda_{i, \text{seq}}^q (f_i(x^l) - \bar{z}_{q,i}) : l = 1, 2, \, i = 1, \ldots, k \)

\[d\] \( \sum_{i=1}^{k} \lambda_{i, \text{seq}}^q = 1 \)

\[e\] \( \lambda_{j, \text{seq}}^q \geq \lambda_{j, \text{seq}}^{q-1} : j = r, \ldots, r+k-1 \)

\[f\] \( \lambda_{r+k, \text{seq}}^q \leq \lambda_{r+k, \text{seq}}^{q-1} \)

where \( M >> N >> \ldots >> 1 \)
Presentation of the method

Step 5

- Use weighting vectors for all possible permutations of weights to restrict the evaluation space.
- Minimization of $\lambda_1$ leads to $[0.0, 0.6, 0.4]$ and $[0.0, 1.0, 0.0]$.
- Minimization of $\lambda_2$ leads to $[0.6, 0.0, 0.4]$ and $[1.0, 0.0, 0.0]$.
- Minimization of $\lambda_3$ leads to $[0.0, 1.0, 0.0]$ and $[1.0, 0.0, 0.0]$.
Step 6

- Form a set of **solutions** that will be considered as potential best choices in the next iteration
- Check proportions between specific evaluations of solution $x$
- Check whether an achievement scalarizing function with a direction determined by solution $x$ compares solutions in the same way as done by the DM
- 39 out of 66 solutions left
Presentation of the method

Step 7

- If the DM feels satisfied with at least one solution found during the process, then the procedure stops.
- If the DM concludes that no compromise point exists or some other stopping criteria are satisfied, then the procedure stops without finding the satisfactory solution.
- If the DM wants to retrack to one of the previous iterations and continue from this point, then go back.
- If the DM wants to continue the solution process, then start new iteration.
Presentation of the method

Second iteration

- Reference point unchanged
- Second comparison:
  \[ x^{41} = [0.20, 0.10, 0.20] \succ
  \succ x^{13} = [0.05, 0.05, 0.40] \]
- Optimize boundary weighting vectors
- Restrict the evaluation space
Presentation of the method

Second iteration

- Only 2 solutions remaining
- Easy to choose the best option

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Summary

- **New interactive approach** for multiple objective optimization problems
- Organization of the **search over the non-dominated set:**
  - Specification of reference points and pairwise comparisons of solutions
  - Incorporation of the preference information in the weights of the achievement scalarizing function
- **All compatible** functions (robust ordinal regression)
- **Intuitiveness**, conviction about what is possible, psychological convergence
Further research

- **Development** of the interactive method:
  - Admitting more diverse preference information
  - Choosing individuals to present for comparison
  - Software with user-friendly interface
  - Case studies

- **Evolutionary interactive cone contraction** MOO method:
  - Integration into an elitist evolutionary multiple objective algorithm, NSGA-II
  - Location of a small set of solutions containing the DM’s ideal option with the highest probability
  - Focus the search and speed up convergence