Optimization Models for Fair Resource Allocation Schemes

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Resource Allocation Problem

- $M = \{1, 2, \ldots, m\}$ – set of services
- Q – set of allocation patterns
- $\mathbf{x} \in Q$ – an allocation pattern
- $y_j = f_j(\mathbf{x})$ – effect of allocation pattern $\mathbf{x}$ on service j

Uniform multicriteria model

$$\max \{[f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})] \; : \; \mathbf{x} \in Q\}$$

Max-Min Solutions and Fairness

- Simple set: $\max\{\min_{j=1,\ldots,m} y_j \ : \ \sum_{j=1}^{m} y_j \leq b \ \} \quad \Rightarrow \quad \bar{y}_j = \frac{b}{m} \ \forall \ j$

- General set: if there exists a nondominated $\bar{\mathbf{y}} \in Y$ satisfying the perfect equity requirement $\bar{y}_1 = \bar{y}_2 = \ldots = \bar{y}_m$, then $\bar{\mathbf{y}}$ is the unique optimal solution of the Max-Min problem on Y.

- Lexicographic Max-Min — Max-Min Fairness (MMF)

$$\text{lexmax } \{(y_{\langle 1\rangle}, y_{\langle 2\rangle}, \ldots, y_{\langle m\rangle}) : \mathbf{y} \in Y\}$$

  where $\langle \mathbf{a} \rangle = (a_{\langle 1\rangle}, a_{\langle 2\rangle}, \ldots, a_{\langle m\rangle})$ the vector obtained from $\mathbf{a}$ by rearranging its components in the non-decreasing order: $a_{\langle 1\rangle} \leq a_{\langle 2\rangle} \leq \ldots \leq a_{\langle m\rangle}$.

Routing design for networks with elastic traffic

Indices: d – demands (pairs), p – allowable paths. e – links

Constants: δ_{edp} – path-link incidence, c_e – link capacity

Decision variables:

flow (bandwidth) allocated to demand d: x_{dp} – on path p, X_d – total

$$\begin{aligned} \textstyle\sum_p x_{dp} &= X_d \quad \forall\, d \\ \textstyle\sum_d \sum_p \delta_{edp} x_{dp} &\leq c_e \quad \forall\, e \\ x_{dp} &\geq 0 \quad \forall\, d,p \end{aligned}$$

Goal: maximize fairly all total flows X_d

Max-Min: maximize the lowest flow — max(min$_d$ X_d)

MMF: having maximized the lowest flow, maximize further the second lowest, etc. — lexmax $(X_{\langle 1\rangle}, X_{\langle 2\rangle}, \ldots, X_{\langle D\rangle})$

Sequential Max-Min Approach

Step 0. Initialize $k = 0$, $Q_0 = Q$ and $M_0 = M$.

Step 1. For the current k solve the Max-Min problem

$$\mathbf{P}_k : \quad z_k = \max_{\mathbf{x}} \{ \min_{j \in M_k} f_j(\mathbf{x}) \ : \ \mathbf{x} \in Q_k \}$$

Step 2. Define $Q_{k+1} = \{\mathbf{x} \in Q_k : \ f_j(\mathbf{x}) \geq z_k \quad \forall \ j \in M_k\}$, identify $B_k = \{j \in M_k : \ f_j(\mathbf{x}) = z_k \quad \forall \ \mathbf{x} \in Q_{k+1}\}$, put $M_{k+1} = M_k - B_k$.

Step 3. If $M_{k+1} = \emptyset$, then go to *Step 4*, otherwise increment k by 1 and return to *Step 1*.

Step 4. Stop. The last set Q_{k+1} is the set of all the MMF solutions.

Sequential Max-Min Algorithms

- For convex programs there exist at least one blocking function f_{j_o} constant on the entire set Q_{k+1} of optimal solutions to $\mathbf{P}_k$ which can be identified with positive dual variables (multipliers) for inequalities $f_j(\mathbf{x}) \geq z$.

- Discrete counter-example

$$\text{lexmax } \{\langle(x_1 + 2x_2, 3x_1 + x_2)\rangle : x_1 + x_2 = 1,\ x_1, x_2 \in \{0, 1\}\}$$

two feasible vectors: $\mathbf{x}^1 = (1, 0)$, $\mathbf{x}^2 = (0, 1)$ with outcomes $\mathbf{y}^1 = (1, 3)$, $\mathbf{y}^2 = (2, 1)$. Obviously, $\mathbf{x}^1$ is the unique MMF solution as $\langle(1, 3)\rangle >_{lex} \langle(2, 1)\rangle$.
Neither f_1 nor f_2 is a blocking function in Max-Min problem

$$\max \{\min\{x_1 + 2x_2, 3x_1 + x_2\} : x_1 + x_2 = 1,\ x_1, x_2 \in \{0, 1\}\}$$

Non-convex (discrete) extensions

- single-path (nonbifurcated) flows

$$\begin{array}{rcll} x_{dp} & \leq & U_d u_{dp} & \forall\ d,p \\ \sum_p u_{dp} & = & 1 & \forall\ d \\ u_{dp} & \in & \{0,1\} & \forall\ d,p \end{array}$$

  assuming U_d upper bound on the largest possible flow X_d

- modularity of the link capacity

$$c_e \quad \leftarrow \quad C y_e, \quad y_e - \text{ integer}$$

  assuming C the capacity (bandwidth) module

Direct discrete formulation

With the use of auxiliary integer variables, any MMF problem (either convex or non-convex) can be formulated as the standard lexicographic maximization with directly defined achievement functions:

$$\begin{array}{cll} \text{lexmax} & (r_1, r_2, \ldots, r_m) & \\ \text{s.t.} & \mathbf{x} \in Q & \\ & r_k - y_j \leq C z_{kj},\ \ z_{kj} \in \{0,1\} & \forall\ j,k \\ & \sum_j z_{kj} \leq k-1 & \forall\ k \end{array}$$

where C is a sufficiently large constant (larger than any possible difference between various individual outcomes y_j).
For $k = 1$ all binary variables z_{1j} are forced to 0 thus reducing the optimization in this case to the standard LP model. For any other $k > 1$ all m binary variables z_{kj} are important.

Conditional means

Cumulated achievements: $\bar{\theta}_k(\mathbf{y}) = \sum_{j=1}^{k} \theta_j(\mathbf{y})$

$$\Theta(\mathbf{y}') \geq_{lex} \Theta(\mathbf{y}'') \quad \Leftrightarrow \quad \bar{\Theta}(\mathbf{y}') \geq_{lex} \bar{\Theta}(\mathbf{y}'').$$

$\mathbf{x} \in Q$ is an optimal solution of the lexicographic problem

$$\text{lexmax } \{(\theta_1(\mathbf{f}(\mathbf{x})), \theta_2(\mathbf{f}(\mathbf{x})), \ldots, \theta_m(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}.$$

iff it is the optimal solution of the cumulated lexicographic problem

$$\text{lexmax } \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}$$

$\theta_j(\mathbf{y}) = y_{\langle j \rangle} \quad \Rightarrow \quad \bar{\theta}_k(\mathbf{y}) = \sum_{j=1}^{k} y_{\langle j \rangle}$

$\mu_k(\mathbf{y}) = \bar{\theta}_k(\mathbf{y})/k$ — conditional mean

Conditional means – computational models

- LP for a given vector $\mathbf{y}$ while nonlinear for variable $\mathbf{y}$:

$$\begin{array}{rcl}\bar{\theta}_k(\mathbf{y}) & = & \min \displaystyle\sum_{j=1}^{m} y_j u_{kj} \\ & & \text{s.t.} \ \displaystyle\sum_{j=1}^{m} u_{kj} = k, \ 0 \le u_{kj} \le 1 \qquad \forall \ j\end{array}$$

- By taking advantage of the LP duality:

$$\begin{array}{rcl}\bar{\theta}_k(\mathbf{y}) & = & \max \left(k r_k - \displaystyle\sum_{j=1}^{m} d_{kj} \right) \\ & & \text{s.t.} \ r_k - y_j \le d_{kj}, \ d_{kj} \ge 0 \qquad \forall \ j\end{array}$$

  LP even for variable $\mathbf{y}$

Direct convex formulation

With the use of conditional means, any MMF problem (either convex or non-convex) can be formulated as the standard lexicographic maximization with directly defined achievement functions:

$$\text{lexmax } \{(\mu_1(\mathbf{f}(\mathbf{x})), \mu_2(\mathbf{f}(\mathbf{x})), \ldots, \mu_m(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\}$$

and implemented without any additional discrete structure:

$$\begin{aligned} \text{lexmax} \quad & (r_1 - \sum_{j=1}^{m} d_{1j},\ r_2 - \frac{1}{2}\sum_{j=1}^{m} d_{2j}, \ldots, r_m - \frac{1}{m}\sum_{j=1}^{m} d_{mj}) \\ \text{s.t.} \quad & \\ & \mathbf{x} \in Q \\ & d_{kj} \geq r_k - f_j(\mathbf{x}), \qquad d_{kj} \geq 0 \qquad \forall\ j,k \end{aligned}$$

Distribution approach

- $V = \{v_1, v_2, \ldots, v_r\}$ $(v_1 < v_2 < \cdots < v_r)$ – all attainable values

- $h_k(\mathbf{y})$ – number of values v_k taken in outcome $\mathbf{y}$ ('pdf')

- $\bar{h}_k(\mathbf{y}) = \sum_{l=1}^{k} h_l(\mathbf{y})$ – number of values smaller or equal v_k ('cdf')

$$\bar{h}_k(\mathbf{y}) = \min\{\sum_{j=1}^{m} z_{kj} \ : \ v_{k+1} - y_j \leq C z_{kj}, \ z_{kj} \in \{0,1\} \quad \forall\ j\}$$

- MMF problem

$$\text{lexmin}\ \{(\bar{h}_1(\mathbf{f}(\mathbf{x})), \bar{h}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{h}_r(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\}$$

Distribution approach

- Taking advantages of possible weighting and cumulating achievements in lexicographic optimization

  $$\widehat{h}_k(\mathbf{y}) = \sum_{l=1}^{k-1} (v_{l+1} - v_l)\bar{h}_l(\mathbf{y}) = \sum_{j=1}^{m} (v_k - y_j)_+ = \sum_{j=1}^{m} \max\{v_k - y_j, 0\}$$

  total (average) shortage to v_k ('integrated cdf')

- MMF problem: lex min $\{(\widehat{h}_1(\mathbf{f}(\mathbf{x})), \ldots, \widehat{h}_r(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}$ modeled by LP expansion

$$\begin{array}{ll} \text{lexmin} & \left[\sum_{j=1}^{m} t_{2j},\ \sum_{j=1}^{m} t_{3j}, \ldots,\ \sum_{j=1}^{m} t_{rj} \right] \\ \text{s.t.} & \mathbf{x} \in Q \\ & v_k - f_j(\mathbf{x}) \leq t_{kj},\ t_{kj} \geq 0 \ \forall\ j, k \end{array}$$

Concluding remarks

- Fair treatment of services can be formalized through the MMF solution concept, which assumes that the worst service performance is maximized and the solution is additionally regularized with the lexicographic maximization of the second worst performance, the third one etc.

- Several efficient sequential Max-Min algorithms can be used to solve convex variants of these problems. The values of dual variables can be used to considerably reduce the number of solved subproblems.

- Sequential Max-Min algorithms are only applicable to convex problems.

Concluding remarks (cont.)

- Any MMF problem can be replaced with the lexicographic maximization of the vector that describes the conditional means, which is an LP expansion of the original problem.

- Any MMF problem can be replaced with the lexicographic minimization of the vector that describes the distribution of outcome values, which can be modeled as as an LP expansion of the original problem.

- The complexity of the distribution model is directly related to the number of different outcome values, which can be artificially limited to a given grid thus creating approximated models. Further research on application of this approach to various classes of MMF problems seems to be very promising.