Title
Solution of Evolutionary Games via Hamilton-Jacobi-Bellman Equations

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Abstract
This poster is focused on construction of solutions for bimatrix evolutionary games based on methods of the theory of optimal control and generalized solutions of Hamilton-Jacobi-Bellman equations. It is assumed that the evolutionary dynamics describe interactions of agents in large population groups in biological and social models or interactions of investors in financial markets.

Interactions of agents are subject to the dynamic process which provides the possibility to control flows between different types of behavior or investments. It is worth noting that the dynamics of interactions can be interpreted as the system of Kolmogorov’s type differential equations. Parameters of the dynamics are not fixed a priori and can be treated as controls constructed either as time programs or on the feedback principle.

Payoff functionals in the evolutionary game of two coalitions are determined by the limit of average matrix gains on an infinite horizon. The notion of a dynamical Nash equilibrium is introduced in the class of control feedbacks within Krasovskii’s theory of differential games.

Elements of a dynamical Nash equilibrium are based on guaranteed feedbacks constructed within the framework of the theory of generalized solutions of Hamilton-Jacobi-Bellman equations. The value functions for the series of differential games are constructed analytically and their stability properties are verified using the technique of conjugate derivatives.

The equilibrium trajectories are generated on the basis of positive feedbacks originated by value functions. It is shown that the proposed approach provides new qualitative results for the equilibrium trajectories in evolutionary games and ensures better results for payoff functionals than replicator dynamics in evolutionary games or Nash values in static bimatrix games.

The efficiency of the proposed approach is demonstrated by applications to construction of equilibrium dynamics for agents’ interactions in financial markets.


**Introduction**

The paper is focused on construction of solution for bimatrix evolutionary games based on the theory of optimal control and generalized solutions of Hamilton-Jacobi-Bellman equations. It is assumed that the evolutionary dynamics describes interactions of agents in large population groups in biological and social models or interactions of investors on financial markets. Interactions of agents are subject to the dynamic process which provides the possibility to control flows between different types of behavior or investments. Parameters of the dynamics are not fixed a priori and can be treated as controls constructed either as time programs or feedbacks. Payoff functionals in the evolutionary game of two coalitions are determined by the limit of average matrix gains on infinite horizon. The notion of a dynamical Nash equilibrium is introduced in the class of control feedbacks within Krasovskii’s theory of differential games. Elements of a dynamical Nash equilibrium are based on guaranteed feedbacks constructed within the framework of the theory of generalized solutions of Hamilton-Jacobi-Bellman equations. The value function for the series of differential games are constructed analytically and their stability properties are verified using the technique of conjugate derivatives. The equilibrium trajectories are generated on the basis of positive feedbacks originated by value functions. It is shown that the proposed approach provides new qualitative results for the equilibrium trajectories in evolutionary games and ensures better results for payoff functionals or replicator dynamics in evolutionary games and Nash values in static bimatrix games. The efficiency of the proposed approach is demonstrated by applications to construction of evolutionary dynamics for agents’ interactions on financial markets.

**Evolutionary Game**

Let us consider the system of differential equations which describes behavioral dynamics for two coalitions:

\[
\begin{align*}
\dot{x} &= -x + u, \\
\dot{y} &= -y + v.
\end{align*}
\]

Parameter \(x, 0 < x < 1\) is the probability of the fact that a randomly taken individual of the first coalition holds the first strategy. Parameter \(y, 0 < y < 1\) is the probability of choosing the first strategy by an individual of the second coalition. Control parameters \(u, v\) and satisfy the restrictions \(0 < u, v < 1\) and can be interpreted as signals for individuals to change their strategies. The system dynamics (1) is interpreted as a version of controlled Kolmogorov’s equation [5] and generalizes evolutionary games dynamics [1, 2, 3, 9]. The term payoff functionals of coalitions are defined as mathematical expectations corresponding to payoff matrices \(A = (a_{ij}), B = (b_{ij})\) (1, 2, 1, 2, 1, 2) and can be interpreted as “local” interests of coalitions:

\[
g_y(j, T) = C_y gj(T) = -a_{ij} gj(T) + a_{ij} T + a_{ij} T = 0.
\]

At a given instant \(T\), here parameters \(C_y, a_{ij}, a_{ij}\) are determined according to the classical theory of bimatrix games [12]:

\[
C_y = a_{ij} = a_{ij} - a_{ij} - a_{ij} - a_{ij} - a_{ij}.
\]

The payoff function \(w_{xy}(\tau)\) for the second coalition is determined according to coefficients of matrix \(B\). “Global” interests \(J_{xy}(\tau)\) of the first coalition are defined as:

\[
J_{xy}(\tau) = \sum_{i=1}^{N} x_i \frac{\partial g_i(j, T)}{\partial y}.
\]

Interests \(J_{xy}(\tau)\) are defined analogously. We consider the solution of the evolutionary game basing on the optimal control theory [10] and differential games [8]. Following [4, 7, 8, 9] we introduce the notion of a dynamical Nash equilibrium in the class of closed-loop strategies (feedbacks) \(u = u(x, y), v = v(x, y)\). Definition 1. Let \(x(0) = x_0, y(0) = y_0, [0, 1] \times [0, 1]\). A pair of feedbacks \(u = u(x, y), v = v(x, y)\) is called a Nash equilibrium for an initial position \(x(0), y(0)\) if for any other feedbacks \(U = u(x, y), V = v(x, y)\), the following condition holds: the inequalities:

\[
J_{xy}(\tau) = \sum_{i=1}^{N} x_i \frac{\partial g_i(j, T)}{\partial y} = \frac{\partial g_i(j, T)}{\partial y} = 0,
\]

are valid for all trajectories:

\[
(x(t), y(t)) \in (x_0, y_0), E(x, y), V(x, y), \text{ and the condition following holds: the inequalities:
\]

\[
J_{xy}(\tau) = \sum_{i=1}^{N} x_i \frac{\partial g_i(j, T)}{\partial y} = \frac{\partial g_i(j, T)}{\partial y} = 0,
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\]

\[
J_{xy}(\tau) = \sum_{i=1}^{N} x_i \frac{\partial g_i(j, T)}{\partial y} = \frac{\partial g_i(j, T)}{\partial y} = 0.
\]

Here the symbol \(\tau\) stands for the set of trajectories, which start from the initial point \((x_0, y_0)\) and are generated by the corresponding strategies \((U^*, V^*), \ (U^*, V^*), \ (U^*, V^*)\). The payoff function can be constructed by pasting positive feedbacks \(u^*, v^*\) and punishing feedbacks \(u^*, v^*\) according to relations (4):

\[
\begin{align*}
J_{xy}(\tau) &= \frac{\partial g_i(j, T)}{\partial y} \quad \text{or otherwise}, \\
J_{xy}(\tau) &= \frac{\partial g_i(j, T)}{\partial y} \quad \text{otherwise}.
\end{align*}
\]

**Value Function for Positive Feedback**

The main role in construction of dynamic Nash equilibrium belongs to positive feedbacks \(w_{xy}^*, v_{xy}^*\), which maximize with guarantee the mean values \(g_y(j, y)\) on the infinite horizon \(T \to \infty\). For this purpose we introduce value functions \(w_{xy}\) in zero-sum games with the infinite horizon. Basing on the method of generalized characteristics for Hamilton-Jacobi-Bellman equations we obtain the analytical structure for value functions. For example, in the case when \(C_y > 0\) the value function \(w_{xy}\) is determined by the system of four functions:

\[
\begin{align*}
w_y(j, T) &= w^*(j, T), \\
w_y(j, T) &= w^*(j, T), \\
w_y(j, T) &= w^*(j, T), \\
w_y(j, T) &= w^*(j, T).
\end{align*}
\]

Here \(w_{xy}\) is the value of the static game with matrix \(A\). The value function \(w_{xy}\) is presented in the Figure 1.

**References**


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