Program packages method for solving closed-loop guidance problem with incomplete information for linear systems

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Let us consider a linear dynamic control system:

\[
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) + c(t), \ t \in [t_0, \vartheta] \\
x(t_0) &= x_0
\end{aligned}
\]  

(1)

\(A(\cdot), B(\cdot), c(\cdot)\) are continuous functions on \([t_0, \vartheta]\);

\(A(t) \in \mathbb{R}^{n \times n}, B(t) \in \mathbb{R}^{n \times m}, c(t) \in \mathbb{R}^n, t \in [t_0, \vartheta]\)

\(u(\cdot)\) is Lebesgue measurable on \([t_0, \vartheta]\), \(u(t) \in P – \text{convex compact}\).

Initial states set \(X_0 \subset \mathbb{R}^n\) is finite, target set \(M \subset \mathbb{R}^n\) is closed and convex.

Signal \(y(t) = Q(t)x(t)\), where \(Q(\cdot) : [t_0, \vartheta] \rightarrow \mathbb{R}^{q \times n}\) - given piecewise continuous matrix-function.
Guaranteed positional guidance problem at pre-defined time

Guaranteed positional guidance problem:

\[ \varepsilon > 0 \]
\[ x_0 \in X_0 \quad \xrightarrow{u(y(\cdot), \cdot))} \quad x(\vartheta | x_0, u(y(\cdot), \cdot)) \in [M]^\varepsilon \]
\[ y(t), t \in [t_0, \vartheta] \]
Homogeneous signals and initial states sets corresponding to homogeneous signals

Let us consider the homogeneous system

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t), \\
x(t_0) &= x_0 \in X_0
\end{align*}
\]

(2)

corresponding to (1). Its solutions is given by the Cauchy formula \(x(t) = F(t, t_0)x_0,\)

\[F(t, t_0) = e^{\int_{t_0}^{t} A(s)ds} \]

Definition

Let us call function \(g_{x_0}(\cdot)\) to be a **homogeneous signal**, corresponding to an admissible initial state \(x_0:\)

\[g_{x_0}(t) = Q(t)F(t, t_0)x_0 \quad (x_0 \in X_0, t \in [t_0, \tau])\]

Set of all admissible initial states \(x_0\), corresponding to the homogeneous signal \(g(\cdot)\) till time-point \(\tau \in [t_0, \tau]:\)

\[X_0(\tau|g(\cdot)) = \{x_0 \in X_0 : g(\cdot)|_{[t_0, \tau]} = g_{x_0}(\cdot)|_{[t_0, \tau]}\};\]
Package guidance problem

Let $G$ be the set of homogeneous signals corresponding to all admissible initial states $x_0 \in X_0$. Let us note that $|G| \leq |X_0|$.

**Non-anticipatory condition:**

$$g(\cdot) \in G$$

$$\tau \in (t_0, \vartheta] \implies u_{x'_0}(t) = u_{x''_0}(t), \ t \in [t_0, \tau]$$

$$x'_0, x''_0 \in X_0(\tau|g(\cdot))$$

**Definition**

Open-loop controls family $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is a program package, if its elements satisfy non-anticipatory condition. Program package $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is guiding, if for every admissible initial state $x_0 \in X_0$ the condition $x(\vartheta|x_0, u_{x_0}(\cdot)) \in M$ holds.

If a guiding program package exists then the package guidance problem is solvable.

**Theorem (Osipov, Kryazhimskiy, 2006)**

The problem of positional guidance is solvable if and only if the problem of package guidance is solvable.
Guaranteed positional guidance problem at pre-defined time solution scheme

- Guaranteed positional guidance problem
  - Osipov, Kryazhimskiy (2006) - general case
  - Package guidance problem
    - Kryazhimskiy, Strelkovskiy (2014) - linear systems
  - Extended program guidance problem
Homogeneous signals split

For an arbitrary homogeneous signal \( g(\cdot) \in G \) let

\[
G_0(g(\cdot)) = \left\{ \tilde{g}(\cdot) \in G : \lim_{\Delta \to +0} |\tilde{g}(t_0 + \Delta) - g(t_0 + \Delta)|_{\mathbb{R}^q} = 0 \right\}
\]

be the set of all homogeneous signals equal to \( g(\cdot) \) in the right-sided neighborhood of the initial time-point \( t_0 \).

**Definition**

Time-point \( \tau_1(g(\cdot)) \) such that for any homogeneous signal \( \tilde{g}(\cdot) \in G_0(g(\cdot)) \), certain homogeneous signal \( g^*(\cdot) \in G_0(g(\cdot)) \) and certain time-point \( \tau_1 < \tau_1^*(g(\cdot)) \leq \vartheta \) the equalities

\[
|\tilde{g}(t) - g(t)|_{\mathbb{R}^q} = 0, \quad t \in [t_0, \tau_1(g(\cdot))] \\
|g^*(t) - g(t)|_{\mathbb{R}^q} > 0, \quad t \in (\tau_1(g(\cdot)), \tau_1^*(g(\cdot))]
\]

hold, is the first **split point** of the homogeneous signal \( g(\cdot) \).
Homogeneous signals split

If $\tau_1(g(\cdot)) < \vartheta$, then for every $k = 2, 3, \ldots$ let

$$G_k(g(\cdot)) = \left\{ \tilde{g}(\cdot) \in G_{k-1}(g(\cdot)) : \lim_{\Delta \to +0} |\tilde{g}(\tau_k(g(\cdot)) + \Delta) - g(\tau_k(g(\cdot)) + \Delta)|_{\mathbb{R}^q} = 0 \right\}$$

be the set of all homogeneous signals from $G_{k-1}(g(\cdot))$, equal to $g(\cdot)$ in right-sided the neighborhood of the time-point $\tau_k(g(\cdot))$.

Definition

Time-point $\tau_{k+1}(g(\cdot))$ such that for any homogeneous signal $\tilde{g}(\cdot) \in G_k(g(\cdot))$, a certain homogeneous signal $g^*(\cdot) \in G_k(g(\cdot))$ and a certain time-point $\tau_{k+1} < \tau_{k+1}^*(g(\cdot)) \leq \vartheta$ the equalities

$$|\tilde{g}(t) - g(t)|_{\mathbb{R}^q} = 0, \ t \in [\tau_k, \tau_{k+1}(g(\cdot))]$$
$$|g^*(t) - g(t)|_{\mathbb{R}^q} > 0, \ t \in (\tau_{k+1}(g(\cdot)), \tau_{k+1}^*(g(\cdot))]$$

hold, is the $(k + 1)$ split point of the homogeneous signal $g(\cdot)$.

Let

$$T(g(\cdot)) = \{\tau_k(g(\cdot)) : k = 1, \ldots, K_{g(\cdot)}\}$$

be the set of all split points of the homogeneous signal $g(\cdot)$ and

$$T = \bigcup_{g(\cdot) \in G} T(g(\cdot)) = \{\tau_1, \ldots, \tau_K\}$$

be the set of all split points of all homogeneous signals. Let us note that $T$ is finite, $|T| = K$ and $|T| \leq |X_0|$.
Initial states set clustering

Definition
For every \( k = 1, \ldots, K \) let the set \( X_0(\tau_k) = \{ X_0(\tau_k | g(\cdot)) : g(\cdot) \in G \} \) be the cluster position at the time-point \( \tau_k \), and each its element – a cluster at this time-point.

Lemma
Open-loop control family \( (u_{x_0}(\cdot))_{x_0 \in X_0} \) is a program package if and only if for every homogeneous signal \( g(\cdot) \in G \), every \( \tau \in T(g(\cdot)) \) and all initial states \( x_0', x_0'' \in X_0(\tau | g(\cdot)) \) the equality \( u_{x_0'}(t) = u_{x_0''}(t) \) holds for all \( t \in [t_0, \tau] \).

Lemma
Programs family \( (u_{x_0}(\cdot))_{x_0 \in X_0} \) is a program package if and only if for every \( k = 1, \ldots, K \), every cluster \( X_{0k} \in X_0(\tau_k) \) and all initial states \( x_0', x_0'' \in X_{0k} \) equation \( u_{x_0'}(t) = u_{x_0''}(t) \) holds for all \( t \in (\tau_{k-1}, \tau_k] \) in case \( k > 1 \) and for all \( t \in [t_0, \tau_1] \) in case \( k = 1 \).
Extended program control

Let $\mathcal{P}$ be the set of all vector families $(u_{x_0})_{x_0 \in X_0}$ such that $u_{x_0} \in P$.

**Definition**

Let us call any measurable function $t \mapsto (u_{x_0}(t))_{x_0 \in X_0} : [t_0, t] \mapsto \mathcal{P}$ to be an extended program.

For each $k = 1, \ldots, K$ let $\mathcal{P}_k$ be the set of all families $(u_{x_0})_{x_0 \in X_0} \in \mathcal{P}$ such that for each cluster $X_{0k} \in \mathcal{X}_0(\tau_k)$ and for all $x_0', x_0'' \in X_{0k}$ holds $u_{x_0'} = u_{x_0''}$.

**Definition**

Extended program control $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is admissible, if for every $k = 1, \ldots, K$ is held $(u_{x_0}(t))_{x_0 \in X_0} \in \mathcal{P}_k$ for all $t \in (\tau_{k-1}, \tau_k]$. We call $\mathcal{P}_k$ to be extended admissible control set on $(\tau_{k-1}, \tau_k]$.

**Lemma**

*Extended program control $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is a control package if and only if it is admissible.*
Homogeneous signals split

Extended program control

Initial states set clustering
Extended space

Let $\mathcal{R}^j$, $j = 1, 2, \ldots$ be finite-dimensional Hilbert space of all vector families $(r_{x_0})_{x_0 \in X_0}$ from $\mathbb{R}^j$ with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ defined as

$$\langle (r'_{x_0})_{x_0 \in X_0}, (r''_{x_0})_{x_0 \in X_0} \rangle_{\mathcal{R}^j} = \sum_{x_0 \in X_0} \langle r'_{x_0}, r''_{x_0} \rangle \quad ((r'_{x_0})_{x_0 \in X_0}, (r''_{x_0})_{x_0 \in X_0} \in \mathcal{R}^j).$$

For every non-empty set $\mathcal{E} \subset \mathcal{R}^j$ let us define lower and upper support functions

$$\rho^-((l_{x_0})_{x_0 \in X_0} | \mathcal{E}) = \inf_{(e_{x_0})_{x_0 \in X_0}} \langle (l_{x_0})_{x_0 \in X_0}, (e_{x_0})_{x_0 \in X_0} \rangle_{\mathcal{R}^j}, ((l_{x_0})_{x_0 \in X_0} \in \mathcal{R}^j)$$

$$\rho^+((l_{x_0})_{x_0 \in X_0} | \mathcal{E}) = \sup_{(e_{x_0})_{x_0 \in X_0}} \langle (l_{x_0})_{x_0 \in X_0}, (e_{x_0})_{x_0 \in X_0} \rangle_{\mathcal{R}^j}, ((l_{x_0})_{x_0 \in X_0} \in \mathcal{R}^j)$$

Lemma

Lower support function of the admissible control set $\mathcal{P}_k$ is given by

$$\rho^-((l_{x_0})_{x_0 \in X_0} | \mathcal{P}_k) = \sum_{x_{0k} \in X_0(\tau_k)} \rho^- \left( \sum_{x_0 \in X_{0k}} l_{x_0} | \mathcal{P} \right), \quad ((l_{x_0})_{x_0 \in X_0} \in \mathcal{R}^m).$$
Let us introduce an extended system, composed of system (1) copies, where each copy is parametrized by elements of $X_0$, namely each $x_0 \in X_0$ is the initial state for the corresponding copy:

$$\dot{x}_{x_0}(t) = A(t)x_{x_0}(t) + B(t)u_{x_0}(t) + c(t), x_{x_0}(t_0) = x_0 \quad (x_0 \in X_0)$$

Let us define the phase state of the extended system (in space $\mathbb{R}^n$) as a family of phase states of the components of the extended system.

**Definition**

Let extended target set $M$ be the set of all families $(z_{x_0})_{x_0 \in X_0} \in \mathbb{R}^n$ such that $z_{x_0} \in M$ for all $x_0 \in X_0$.

**Definition**

An admissible extended program $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is guiding the extended system, if $(x(\cdot|0, u_{x_0}(\cdot)))_{x_0 \in X_0} \in M$.

The extended problem of program guidance is solvable, if there exists an admissible extended program which is guiding the extended system.
Theorem (Kryazhimskiy, Strelkovskiy, 2014)

1) The package guidance problem is solvable if and only if the extended program guidance problem is solvable. 2) An admissible extended program is a guiding program package if and only if it is guiding extended system.

For every \((l_{x_0})_{x_0} \in \mathcal{X}_0 \in \mathbb{R}^n\) and \(a \in [0, 1]\) let us define the function

\[
\gamma_a((l_{x_0})_{x_0} \in \mathcal{X}_0) = \rho^-(((l_{x_0})_{x_0} \in \mathcal{X}_0 | \mathcal{A}) - \rho^+((l_{x_0})_{x_0} \in \mathcal{X}_0 | \mathcal{M}) = \sum_{x_0 \in \mathcal{X}_0} \langle l_{x_0}, F(\vartheta, t_0)x_0 \rangle_{\mathbb{R}^n} + \\
\sum_{k=1}^{K} \int_{\tau_{k-1}}^{\tau_k} \sum_{x_0k \in \mathcal{X}_0(\tau_k)} \rho^-(\sum_{x_0 \in \mathcal{X}_0k} B^T(s)F^T(\vartheta, s)l_{x_0} | aP) ds + \\
\int_{t_0}^{\vartheta} \left\langle \sum_{x_0 \in \mathcal{X}_0} l_{x_0} , F(\vartheta, s)c(s) \right\rangle_{\mathbb{R}^n} ds - \sum_{x_0 \in \mathcal{X}_0} \rho^+(l_{x_0} | \mathcal{M})
\]

Theorem (Kryazhimskiy, Strelkovskiy, 2014)

Each of the three problems – (i) the extended program guidance problem, (ii) the package guidance problem and (iii) the guaranteed positional guidance problem – is solvable if and only if

\[
\sup_{(l_{x_0})_{x_0} \in \mathcal{X}_0 \in \mathbb{R}^n} \gamma_1((l_{x_0})_{x_0} \in \mathcal{X}_0) \leq 0.
\]
Construction of the guiding program package

**Definition**

Let the program package \((u^0_{x_0}(\cdot))_{x_0 \in X_0}\) be **zero-valued**, if \(u^0_{x_0}(t) = 0\) for almost all \(t \in [t_0, \vartheta]\), \(x_0 \in X_0\).

Zero-valued program package \((u^0_{x_0}(\cdot))_{x_0 \in X_0}\) is guiding if and only if

\[
(x(\vartheta|_{x_0}, u^0_{x_0}(\cdot)))_{x_0 \in X_0} = (F(\vartheta, t_0)_{x_0} + c(\vartheta))_{x_0 \in X_0} \in \mathcal{M}.
\]

**Lemma**

*Let the condition (3) be satisfied, and the zero-valued program package \((u^0_{x_0}(\cdot))_{x_0 \in X_0}\) is not guiding extended system. Then \(a_* \in (0, 1]\) exists such that*

\[
\max_{(l_{x_0})_{x_0 \in X_0} \in \mathcal{L}} \gamma_{a_*}(((l_{x_0})_{x_0 \in X_0})) = 0.
\]
Separation of two nonempty closed convex sets by hyperplane
Theorem (Strelkovskiy, 2015)

Let the set $P$ be strictly convex compact, $0 \in \text{int } P$, and for all $t \in [0, \vartheta]$ matrix $D(t) = B^T(t)F^T(\vartheta, t)$ is full-rank. Let the condition (3), holds and the vector family $(l_{x_0}^*)_{x_0 \in X_0} \in \mathcal{L}$ is maximizing (4). Let the program package $(u_{x_0}^*(\cdot))_{x_0 \in X_0}$ satisfy $u_{x_0}^*(t) \in a_* P$ ($x_0 \in X_0$, $t \in [t_0, \vartheta]$), where $a_*$ is the root of equation (4), and for every $k = 1, \ldots, K$ and all clusters $X_{0k} \in \mathcal{X}(\tau_k)$ the equality

$$\left\langle D(t) \sum_{x_0 \in X_{0k}} l_{x_0}^*, u_{X_{0k}}^*(t) \right\rangle = \rho^- \left( D(t) \sum_{x_0 \in X_{0k}} l_{x_0}^* \bigg\vert a_* P \right) \quad (t \in [\tau_{k-1}, \tau_k])$$

(5)

holds. If for every $k = 1, \ldots, K$ and $X_{0k} \in \mathcal{X}(\tau_k)$ it holds that $\sum_{x_0 \in X_{0k}} l_{x_0}^* \neq 0$, then the program package $(u_{x_0}^*(\cdot))_{x_0 \in X_0}$ is guiding.
Construction of the guiding positional strategy

Let $\mathcal{U}_t = \{u(\cdot)|_{[t_0, t]}, u(\cdot) \in \mathcal{U}, t \in (t_0, \theta]\}$ be the set of programs till the time-point $t$. Let us formally define $\mathcal{U}_{t_0}$ as an arbitrary one-element subset of $P$. Let $\mathcal{Y}_t$ be the set of all functions $y(\cdot) : [t_0, t] \mapsto \mathbb{R}^q$; let us call its elements to be observations till the time-point $t$.

**Definition**

Let us formally define a **positional strategy** as an arbitrary finite sequence

$$S = (\sigma_i, U_i)^R_{i=1},$$

where $t_0 < \sigma_1 < \ldots < \sigma_R = \vartheta$ and $U_i$ is a map of the product $\mathcal{Y}_{\sigma_i} \times \mathcal{U}_{\sigma_i}$ to the resource set $P$ for each $i = 1, \ldots, R$. Moments $\sigma_i$, $i = 1, \ldots, R - 1$ will be called (control) **adjustment moments**, and a map $U_i$ is called a **feedback** at the moment $\sigma_i$.

Positional strategy $S$ is $\varepsilon$-guiding, if for all $\varepsilon > 0$ and $x_0 \in X_0$ the motion $x(\cdot)$ in the controlled process under the action of $S$ satisfies the condition $x(\vartheta) \in [M]^\varepsilon$, $[M]^\varepsilon = M + S_\varepsilon(0)$, where $S_\varepsilon(0)$ is a ball of radius $\varepsilon$ in the space $\mathbb{R}^n$ with the center in zero.
Let us constructively define the guiding positional strategy $S^*$. Let us define *adjustment moments* as

$$
\sigma_k = \begin{cases} 
\delta, & k = 0 \\
\tau_k + \delta, & k = 1, \ldots, K 
\end{cases}
$$

At the segment $[t_0, \sigma_1]$ let us apply an arbitrary constant control $\tilde{u}(t) = \bar{u} \in P$ to the system (1), and at the adjustment moments will adapt the control according to the following rule:

$$
\tilde{u}(t) = u^*_{X_{0k}}(t), \ t \in (\sigma_{k-1}, \sigma_k], \ X_{0k} = X_0(\tau_k | g(\cdot)), \ k = 1, \ldots, K
$$

Here $g(\cdot)$ is the observed homogeneous signal at the segment $[t_0, \sigma_k]$. 

![Diagram](image.png)
Lemma

Let the program package guidance problem be solvable for the system (1), and $\bar{x}(\cdot) = x(\cdot|\bar{x}_0, \bar{u}_x(\cdot))$ is the motion from the initial admissible state $x_0 \in X_0$ under the action of program $\bar{u}_x(\cdot)$ from the guiding program package $\bar{u}_x(\cdot) x_0 \in X_0$. Let $\tilde{x}(\cdot) = x(\cdot|\tilde{x}_0, \tilde{u}(\cdot))$ be the motion in the controlled process $(\tilde{x}(\cdot), y(\cdot), \tilde{u}(\cdot))$ with the initial state $x_0 \in X_0$ under the action of positional strategy $S = (\sigma_k, U_k)_{k=1}^K$, $\tilde{u}(\cdot) \in U$, $y(t) = Q(t)\tilde{x}(t)$ ($t \in [t_0, \varphi]$) and for each $i = 1, \ldots, R - 1$ holds

$$\tilde{u}(t) = \tilde{u}(\sigma_k) = U_i(y_{\sigma_k}(\cdot), u_{\sigma_k}(\cdot)) \quad (t \in [\sigma_k, \sigma_{k+1}]),$$

where $u_{\sigma_i}(\cdot)$ is restriction of $u(\cdot)$ to $[t_0, \sigma_k)$ and $y_{\sigma_k}(\cdot)$ is restriction of $y(\cdot)$ to $[t_0, \sigma_i]$. Then

$$|\tilde{x}(t) - \tilde{x}(t)| \leq KC\delta \quad (t \in [t_0, \varphi]),$$

where $C$ is a positive constant.

Theorem (Strelkovskiy, 2015)

Let the program package guidance problem be solvable for the system (1), and condition $KC\delta < \varepsilon$ holds, where $\delta > 0$, and $C$ is a certain positive constant. Then the positional strategy $S = (\sigma_k, U_k)_{k=1}^K$, corresponding to the guiding program package, is $\varepsilon$-guiding.
Example

Let us consider a dynamic controlled system

\[
\begin{align*}
\dot{x}_1 &= x_2, \quad x_1(0) = x_{01} \\
\dot{x}_2 &= u, \quad x_2(0) = x_{02}
\end{align*}
\]

on the time segment \([0, 2]\).

\[
X_0 = \{ x'_0, x''_0 \}; \quad M = \left\{ \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbb{R}^2 : |x_1| \leq 1, x_2 \in \mathbb{R} \right\}
\]

\[
u(t) \in P = \{ u : |u| \leq 1 \}, \quad t \in [0, 2]; \quad Q(t) = \left\{ \begin{array}{ll}
(0, 0), & t \in [0, 1] \\
(1, 0), & t \in (1, 2].
\end{array} \right\}
\]
Example

Solving maximization problem (3) we get

\[ \gamma_1 \approx -0.35 \quad < \quad 0 \Rightarrow \text{The package guidance problem is solvable} \]

\[ \gamma a_* = 0, \quad a_* = \frac{1}{2} \]

From the theorem (5) it is clear that the guiding program package is

\[
\begin{cases}
  u^*_{x_0 t x_0'}(t) = \frac{1}{2}, & t \in [0, 1], \\
  u^*_{x_0 t}(t) = \frac{1}{2}, & t \in (1, 2], \\
  u^*_{x_0 t'}(t) = -\frac{1}{2}, & t \in (1, 2].
\end{cases}
\]


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