ON THE VALUE OF POPULATION IN DISTRIBUTED CONTROL MODELS

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Motivation I

- Research and management regularly assign values to populations as well as to individual members of a population:
  - Demography / evolutionary biology: Reproductive value, evolutionary fitness
  - Resource management: Hotelling rule, Hartwick rule, etc. assign values (e.g. of instantaneous harvest as opposed to future harvest)
  - Environmental management (i): use value + option value + non-use value
  - Environmental management (ii): Environmental impact of humans (e.g. carbon footprint)
  - Business management: capital stock, inventory
Motivation 2

Valuing human life:

“Economists know the price of everything but the value of nothing” (Oscar Wilde)

- Value of a statistical life in medicine, health and environmental economics, occupational safety, public policy
- Human capital
- Valuing birth control (in terms of development impacts)
- Valuing environmental impact of a person (e.g. carbon footprint)
Motivation 3

Conceptually:

- Valuations can be implicit (shadow price of population stock) or...
- …explicit, e.g. the reproductive value (measured in numbers of offspring) or the statistical value of life (measured in money)

Our contribution (Wrzaczek et al., Theoretical Population Biology, 2010):

- Formal link between implicit valuations in distributed control models and (economic) valuations
- Applications to health economics, epidemiology, resource management in a predator-prey-setting.
Basics

Population decrement (McKendrick equation):

\[ N_a + N_t = -\mu(a, t)N(a, t) \quad \quad N(0, t) = B(t), N(a, 0) = N_0(a) \]

Population renewal:

\[ B(t) = \int_0^\infty \nu(a, t)N(a, t) \, da \]

fertility at age s

(Remaining) Reproductive value (Fisher, 1930):

\[ \psi^R(a, t) = \int_0^\infty e^{-n(s-a)} \frac{l(s, t-a+s)}{l(a, t)} \nu(s, t-a+s) \, ds \]

Rate of population growth survival from a to s
Application 1: Survival investments vs. consumption

- Planner maximizes social welfare of an aged structured population over time by choosing consumption \( c(a,t) \) and survival investments \( h(a,t) \)

- Optimal allocation rule for \( h(a,t) \) embraces the statistical value of life amended by a value of progeny (Kuhn et al. 2010)

**State dynamics:**

For Population:

\[
\left( \frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) N(a, t) = -\mu(a, h(a, t)) N(a, t) \quad N(0, t) = B(t), \quad N(a, 0) = N_0(a)
\]

For Wealth:

\[
\left( \frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) A(a, t) = rA(a, t) + (y(a) - c(a, t) - h(a, t)) N(a, t) \quad A(0, t) = A(\omega, t) = 0 \quad \forall t
\]

\[
A(a, t) = A_0(a), \quad A(a, T) = A_T(a) \quad \forall a,
\]
Application 1: Survival investments vs. consumption

Social welfare objective:

\[
\max_{c(a,t), h(a,t)} \int_0^T \int_0^\omega e^{-\rho t} u(c(a,t)) N(a,t) \, da \, dt
\]

s.t. population and wealth dynamics + boundary conditions

Value of the population state:

\[
\xi^N(a, t) = \int_a^\omega e^{-\rho(s-a)} \frac{l(s, t - a + s)}{l(a, t)} \left[ u(c) + u_c(c)(y - c - h) \right] \, ds
\]

\[
+ \int_a^\omega e^{-\rho(s-a)} \frac{l(s, t - a + s)}{l(a, t)} \nu(s, t - a + s) \xi^N(0, t - a + s) \, ds
\]

Direct "value"  
Productive value  
(Remaining) reproductive value
Application 1: Survival investments vs. consumption

First-order conditions on controls:

\[ \mathcal{H}_c = u_c(c)N - \xi^A N = 0 \quad \text{consumption smoothing across age-groups} \]

\[ \mathcal{H}_h = -\xi^N \mu_h(a, h)N - \xi^A N = 0 \quad \text{optimal survival investments} \]

or in economic terms:

\[ -\frac{1}{\mu_h(a, h)} = \frac{\xi^N}{\xi^A} = \psi^N(a, t) \]

\[ \text{effective \$ cost of saving one life} \quad \text{monetary value of a life saved} \]
Application 1: Survival investments vs. consumption

Monetary value of survival

\[
\psi^N(a, t) = \int_a^\omega e^{-r(s-a)} \frac{l(s, t-a+s)}{l(a, t)} \left[ \frac{u(c)}{u_c(c)} + (y - c - h) \right] ds
\]

\[
+ \int_a^\omega e^{-r(s-a)} \frac{l(s, t-a+s)}{l(a, t)} \nu(s, t-a+s) \frac{u_c(c(0, t-a+s))}{u_c(c(s, t-a+s))} \psi^N(0, t-a+s) ds
\]

\$ Value of a statistical life (as e.g. in Shepard & Zeckhauser, 1984, MSI

Consumer surplus =\$ value of utility

\$ value of a birth at age s

Conversion from „newborn“ values into survivor‘s values (at age s)
Application 1: Survival investments vs. consumption

Value of a birth (value of progeny)

\[ \psi^N(0, t) = \int_0^\omega e^{-rs}l(s, t + s) \left[ \frac{u(c)}{u_c(c)} \right] ds + \int_0^\omega e^{-rs}l(s, t + s)\nu(s, t + s) \frac{u_c(c(0, t + s))}{u_c(c(s, t + s))} \psi^N(0, t + s) ds \]

$\psi^N(0, t)$ = $\psi^N(0, t + \omega) \cdot e^{-r\omega} \left[ \frac{u(c)}{u_c(c)} \right]$

For a stable population & steady state economy:

\[ \psi^N(0, t) = \int_0^\omega e^{-rs}l(s, t + s) \left[ \frac{u(c)}{u_c(c)} \right] ds \]

\[ \int_0^\omega \left[ e^{(r-n)s} - 1 \right] e^{-rs}l(s, t + s)\nu(s, t - a + s) ds \in [0, \infty) \iff r > n \]

Note: zero productive value if life-cycle budget is balanced.

Implies and is implied by dynamic efficiency.

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Application II: Management of a predator-prey system

- Two interacting animal populations: predators (=wolves) and prey (=sheep)
- Optimal culling policies \( u(a,t) \) aimed at wolves and \( w(a,t) \) aimed at sheep
- Predator-prey interaction driven by age-specific hunting effectiveness \( f(a) \) and vulnerability \( g(a) \)

...translating into:

\[
P(t) = \int_0^\omega g(a)B(a, t)da \quad \text{Pool of „effective prey“}
\]
\[
Q(t) = \int_0^\omega f(a)R(a, t)da \quad \text{Predatory pressure}
\]
Application II: Management of a predator-prey system

Population dynamics

**Wolves:**
- natural mortality
- human culling at effort $u(a,t)$

\[ \left( \frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) R(a, t) = -\mu_R(a)R(a, t) - h_R(a, u(a, t)) \]

\[ R(0, t) = G(t) = \int_{0}^{\omega} \nu_R(a)R(a, t)f(a)P(t) \, da, \quad R(a, 0) = R_0(a) \]

Surviving offspring for a mom aged $a$ increases in her hunting success $f(a)P(t)$

**Sheep:**
- natural + wolve-induced mortality depending on „predatory risk“ $g(a)Q(t)$
- human culling at effort $w(a,t)$

\[ \left( \frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) B(a, t) = -\mu_B(a)B(a, t)g(a)Q(t) - h_B(a, w(a, t)) \]

\[ B(0, t) = H(t) = \int_{0}^{\omega} \nu_B(a)B(a, t)da, \quad B(a, 0) = B_0(a) \]
Application II: Management of a predator-prey system

Objective: Net present $ value of the system

$$\max_{u(a,t),w(a,t)} \int_0^T \int_0^\omega e^{-\rho t} F(R(a,t),B(a,t),u(a,t),w(a,t)) \, da \, dt$$

s.t. predator-prey dynamics + boundary conditions

First-order conditions on controls:

$$H_u = F_u - \xi^R(a,t) \frac{\partial h_R(a,u(a,t))}{\partial u(a,t)} = 0$$

$$H_w = F_w - \xi^B(a,t) \frac{\partial h_B(a,w(a,t))}{\partial w(a,t)} = 0$$

marginal return (through sale of culled animals)
– marginal cost of culling
Application II: Management of a predator-prey system

Value of a predator (wolf) aged a at time t

\[ \xi^R(a, t) = \int_a^\omega e^{-\int_a^s[\rho + \mu_R(s')]ds'} \left(F_R(\cdot) \right) \]

Direct value of predator (eco-system value?) \( \geq 0 \)

\[ -f(s) \int_0^\omega B(s', t-a+s)\xi^B(s', t-a+s)ds' \]

Value of the predator’s offspring: \(<0\) for \(F_R=0\)

Loss of prey animals valued at \(\xi^B>0\) due to increased predatory pressure: \(<0\)
Application II: Management of a predator-prey system

Value of a prey animal (sheep) aged $a$ at time $t$

$$\xi^B(a, t) = \int_a^\omega e^{-\int_a^{s'} [\rho + \mu_\mathcal{B}(s')g(s')Q(t-a+s')]ds'} \left( F_B(\cdot) \right)$$

$$+ g(s) \int_0^\omega \nu_R(s') f(s') R(s', t-a+s)\xi^R(0, t-a+s)ds' \right) ds$$

$$+ \int_a^\omega e^{-\int_a^{s'} [\rho + \mu_\mathcal{B}(s')g(s')Q(t-a+s')]ds'} \xi^B(0, t-a+s)\nu_B(s)ds$$

Value of the prey’s offspring: $>0$ for $F_B>0$ (suff. large)

Direct use value of prey (e.g. value of wool, milk, etc.) + option value of future cull $>0$

Support of predator offspring valued at $\xi^R<0$ due to greater pool of prey: $<0$
Application III: HIV prevention

- Two populations: susceptibles (S) and infected (I)

- Susceptible age group $S(a,t)$ interacts with infected at an age-specific rate

\[
P(a, t) = \int_0^\omega \lambda(a, a') \frac{I(a', t)}{S(a', t) + I(a', t)} \, da'
\]

- with $\gamma(a) = \text{base risk of infection}$ and $\phi(u(a,t)) \leq 1$ the impact of an prevention effort $u(a,t)$ the number of newly infected amongst age group $a$ at time $t$ is given by

\[
\gamma(a)\phi(u(a, t))P(a, t)S(a, t)
\]
Application III: HIV prevention

Population dynamics

- **Susceptibles:**
  \[
  \left( \frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) S(a,t) = -\mu_S(a)S(a,t) - \gamma(a)\phi(u(a,t))P(a,t)S(a,t)
  \]
  \[
  S(0,t) = B(t) + \alpha C(t), \quad S(a,0) = S_0(a)
  \]

  - Births by susceptibles
  - Non-infected births by the infected

- **Infected:**
  \[
  \left( \frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) I(a,t) = -\mu_I(a)I(a,t) + \gamma(a)\phi(u(a,t))P(a,t)S(a,t)
  \]
  \[
  I(0,t) = (1 - \alpha)C(t), \quad I(a,0) = I_0(a)
  \]

- with
  \[
  B(t) = \int_0^\omega \nu_S(a,t)S(a,t)\,da
  \]
  \[
  C(t) = \int_0^\omega \nu_I(a,t)I(a,t)\,da
  \]

  - Deaths from infection (or other)
  - New infections
  - Infected births by the infected
Application III: HIV prevention

Objective: Minimise the total cost of infection plus prevention

\[
\max_{u(a,t)} - \int_0^T \int_0^\omega e^{-\rho t} \left( F(I(a,t)) + u(a,t) \right) \text{d}a\text{d}t
\]

- Cost of infection related to age group \(a\) at time \(t\)
- Prevention cost related to age group \(a\) at time \(t\)

First-order condition on control:

\[
\mathcal{H}_u = -1 - \gamma(a) P(t) S(a,t) \frac{\partial \phi}{\partial u} \left[ \xi^S(a,t) - \xi^I(a,t) \right] = 0
\]

- Marginal $ spent on prevention
- Marginal number of infections prevented
- Value of averting an infection
Application III: HIV prevention

Value of a susceptible (in terms of HIV-related costs)

\[ \xi^S(a, t) = \int_a^\omega e^{-\int_a^s \left[ \rho + \mu_S(s') + \gamma\phi(u)P \right] ds'} \left[ \gamma\phi(u)P\xi^I - \int_0^\omega \zeta(t, a') \frac{\lambda I}{(S + I)^2} da' \right] ds \]

\[ + \int_a^\omega e^{-\int_a^s \left[ \rho + \mu_S(s') + \gamma\phi(u)P \right] ds'} \xi^S(0, t - a + s) \nu_S(s) ds \]

Expected value of becoming infected (<0)  
Value in lowering disease prevalence and infection risk: typically >0

Value of non-infected children
Application III: HIV prevention

Value of an infected (in terms of HIV-related costs)

\[ \xi^I(a, t) = \int_a^\omega e^{-\int_a^s [\rho_1 + \mu_1(s')] ds'} \left[ -F_I(I) + \int_0^\omega \zeta(t, a') \frac{\lambda S}{(S + I)^2} da' \right] ds \\
+ \int_a^\omega e^{-\int_a^s [\rho_1 + \mu_1(s')] ds'} \left( \alpha \xi^S(0, t - a + s) + (1 - \alpha) \xi^I(0, t - a + s) \right) \nu_I(s) ds \]

Contribution of an additional infected to the cost of infection

Expected value of non-infected children

Expected value of infected children <0 if \( \alpha \rightarrow 0 \)

Value of raising disease prevalence and infection risk: typically <0
Conclusion

- Shadow prices on population states can be interpreted as values of population. These could be directly in $ terms if the objective function is in $, or they could be converted into $ terms, also in case of the value of human survival.

- As such they can be used to
  
  (a) compare the valuations in the model against the data (to check e.g. if components of the model are missing)
  (b) to calibrate the model to the data
  (c) to provide policy guidance.

- An analysis of the components of the shadow price allows to identify components relevant to the valuation (and the mechanisms behind the model)

- In age-structured models of long-term development of the population, the value of population falls into a „direct“ part relating to the contemporary individual/cohort, and an indirect relating to the value of the individuals expected offspring

- The value of population is age-specific in a non-trivial way
Appendix: General model

\[
\max_{u,v,w} \quad \int_0^T \int_0^\omega e^{-\rho t} L(a, t, N, Y, Q, P, u, v, w) \, da \, dt + \int_0^\omega e^{-\rho T} l(a, Y(a, t)) \, da
\]

s.t.

\[
\left( \frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) N(a, t) = -\mu(a, t, N, Y, Q, P, u)N + g(a, t, N, Y, Q, P, u)
\]

\[
N(0, t) = B(t) = \int_0^\omega \nu(a, t, N, Y, Q, P, u)N \, da, \quad N(a, 0) = N_0(a)
\]

\[
\left( \frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) Y(a, t) = f(a, t, N, Y, Q, P, u)
\]

\[
Y(0, t) = \varphi(t, B, Q, v), \quad Y(a, 0) = Y_0(a, w)
\]

\[
Q(t) = \int_0^\omega h(a, t, N, Y, Q, P, u) \, da
\]

\[
P(a, t) = \int_0^\omega k(a, t, a', N, Y, u) \, da'
\]

\[
u(a, t) \in U, \quad v(t) \in V, \quad w(a) \in W
\]

Salvage value related to non-pop states

Other population change

Density dependent mortality and fertility

Non-population states

Boundary states

Interaction states

Age-structured controls

Boundary controls

Initial controls
Appendix: General model

The distributed, initial and boundary Hamiltonian of the general model reads as follows

\[
\mathcal{H}(\cdot) = L + \xi^N(-\mu N + g) + \xi^Y f + \eta^B \nu N + \eta^Q h + \int_0^\omega \zeta k(a, t, a', u) \, da'
\]

\[
\mathcal{H}_0(\cdot) = \xi^N(a, 0)N_0(a) + \xi^Y(a, 0)Y_0(a, w) + \int_0^T L(a, t, w) \, dt
\]

\[
\mathcal{H}_b(\cdot) = \xi^N(0, t)B(t) + \xi^Y(0, t)\varphi(t, v) + \int_0^\omega L(a, t, v) \, da.
\]  \hspace{1cm} (1)

Applying distributed optimal control theory (see Feichtinger et al. (2003)) we obtain the following adjoint system

\[
\xi^N_a + \xi^N_t = (\rho + \mu + \mu_N N)\xi^N - L_N - \xi^N g_N - \xi^Y f_N - \eta^B (\nu N + \nu) - \eta^Q h_N - \int_0^\omega \zeta^P k_N \, da'
\]

\[
\xi^Y_a + \xi^Y_t = (\rho + f_Y)\xi^Y - L_Y + \xi^N \mu Y N - \xi^N g_Y - \eta^B \nu Y N - \eta^Q h_Y - \int_0^\omega \zeta^P k_Y \, da'
\]

\[
\eta^B = \xi^N(0, t) + \xi^Y(0, t)\varphi_B
\]

\[
\eta^Q = \xi^Y(0, t)\varphi_Q + \int_0^\omega L_Q + \xi^N(-\mu Q N + g_Q) + \xi^Y f_Q + \eta^B \nu Q N + \eta^Q h_Q \, da
\]

\[
\zeta^P = L_P + \xi^N(-\mu_P N + g_P) + \xi^Y f_P + \eta^B \nu P N + \eta^Q h_P
\]
Appendix: General model

together with the transversality conditions

\[ \xi^N(a, T) = 0 \quad \xi^N(\omega, t) = 0 \]
\[ \xi^Y(a, T) = l_Y(a, T) \quad \xi^Y(\omega, t) = 0 \]

Finally the necessary first order conditions can be derived from

\[ \mathcal{H}(a, t, u^*(a, t)) \geq \mathcal{H}(a, t, u(a, t)) \quad \forall \ u(a, t) \in U \]
\[ \frac{\partial \mathcal{H}_0}{\partial w}(a_0, w^*(a_0))(w - w^*(a_0)) \leq 0 \quad \forall \ w \in W \]
\[ \frac{\partial \mathcal{H}_0}{\partial v}(t_0, v^*(t_0))(v - v^*(t_0)) \leq 0 \quad \forall \ v \in V \]

where \( u^*(a, t) \) denotes the distributed, \( w^*(a_0) \) the initial and the \( v^*(t_0) \) the boundary optimal control.
Appendix: General model

\[
\left( \frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) \xi_N(a, t) = \left( \rho + \mu + \mu_N N - g_N \right) \xi_N - L_N - \xi_Y f_N - \\
\quad -\xi_N(0, t - a + s)(\nu_N N + \nu) - \eta^Q h_N - \int_0^\omega \zeta P_N \, d\omega
\]

where \( \xi^Y(a, t) \), \( \eta^Q(t) \) and \( \zeta(a, t) \) are the adjoint variables of \( Y \), \( Q \) and \( P \) respectively. All adjoint variables can be interpreted as dynamic shadow prices, i.e. they indicate the increase of the objective function if the corresponding state is increased marginally. E.g. \( \xi_N(a, t) \) denotes the increase of the objective function if the population is increased marginally at age \( a \) at time \( t \) (or by one \( a \)-aged individual at \( t \) if the population is large enough). The term shadow price has already been used in the examples.

Together with the transversality condition \( \xi_N(\omega, t) = 0 \) the shadow price of the population can be solved with the method of characteristics for all cohorts whose maximal life horizon ends before the planning horizon \( T \)

\[
\xi_N(a, t) = \int_a^\omega e^{-\int_a^s (\rho + \mu + \mu_N N - g_N) \, ds'} \left( L_N + \xi^Y f_N + \eta^Q h_N + \int_0^\omega \zeta P_N \, da' \right) \, ds' + \\
\quad + \int_a^\omega e^{-\int_a^s (\rho + \mu + \mu_N N - g_N) \, ds'} \xi_N(0, t - a + s)(\nu + \nu_N N) \, ds.
\]  \tag{3}
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THANK YOU!