Towards Climate-Economy Assessment with Stochastic Discount Rates

Timm Faulwasser\(^1\), Chris Kellett\(^2\), Steven Weller\(^2\), Willi Semmler\(^3\), Lars Grüne\(^4\)

\(^1\) Institute for Applied Informatics, Karlsruhe Institute of Technology, Germany
\(^2\) School of Electrical Engineering and Computing, University of Newcastle, Australia
\(^3\) New School of Social Research, New York, USA
\(^4\) Chair for Applied Mathematics, University of Bayreuth, Germany

IIASA Summer Workshop on Green Growth Modelling
July 26 2017
Complete DICE model

\[
\max_{\mu(k), s(k)} W = \sum_{k=0}^{N} (1 + \rho)^{-k} U(k)
\]

subject to, for all \( k = 0, \ldots, N - 1 \)

\[
\begin{bmatrix}
M(k+1) \\
T(k+1) \\
K(k+1)
\end{bmatrix} =
\begin{bmatrix}
\Phi & 0 & 0 \\
0 & \Psi & 0 \\
0 & 0 & (1 - \delta)^5
\end{bmatrix}
\begin{bmatrix}
M(k) \\
T(k) \\
K(k)
\end{bmatrix} + 5
\begin{bmatrix}
B_{ME}(K(k), \mu(k), d(k)) \\
B_{TF}(M_{AT}(k), d(k)) \\
f(K(k), T_{AT}(k), s(k), \mu(k), d(k))
\end{bmatrix}
\]

\[
[M(0), T(0), K(0)]^T = [M_0, T_0, K_0]^T
\]

0 \leq [M(k), T(k), K(k)]^T

0 \leq \mu(k) \leq 1

0 \leq s(k) \leq 1

2 \geq T_{AT}(k)

How to account for uncertainties in DICE?

- Feedback policy
- Robustified or stochastic computations
Climate-economy assessment via optimization

\[
\text{maximize} \quad W = \sum_{k=0}^{N} (1 + \rho)^{-k} \cdot U(k)
\]

subject to
- dynamics of global carbon cycle
- global temperature dynamics
- model of the economy
- couplings + constraints

**Discount rate** \( \rho = 0.015 \)

**Discount rate** \( \rho = 0.03 \)

What is the correct discount rate?
Outline

Discount rates in DICE?

Computing the Social Cost of Carbon

How to do computations with stochastic uncertainties?

First results for Polynomial Chaos Expansions in DICE 2013

Summary & outlook
Social Cost of Carbon (SCC)

[The] social cost of carbon in a particular year is the decrease in aggregate consumption in that year that would change the current [...] value of social welfare by the same amount as a one unit increase in carbon emissions [1tCO₂] in that year.


\[
SCC(k) = - \frac{\text{marginal impact of emissions on welfare}}{\text{marginal impact of consumption on welfare}} = - \frac{\partial W}{\partial E(k)} \cdot \frac{\partial E(k)}{\partial C(k)}
\]
How to compute the SCC?

Code the NLP

$$\max_{\mu(k), s(k)} \quad W = \sum_{k=0}^{N} (1 + \rho)^{-k} U(k)$$

subject to

- dynamics of global carbon cycle
- global temperature dynamics
- model of the economy
- couplings + constraints

such that the equality constraints

$$E(k) = \ldots$$
$$C(k) = \ldots$$

appear explicitly

$$\Rightarrow \quad \frac{\partial W}{\partial E(k)} \quad \text{and} \quad \frac{\partial W}{\partial C(k)} = \text{corresponding Lagrange multipliers}$$

Uncertain discount rates in DICE

\[
\begin{align*}
\text{maximize} & \quad W = \sum_{k=0}^{N} (1 + \rho)^{-k} \cdot U(k) \\
\text{subject to} & \quad \text{dynamics of global carbon cycle} \\
& \quad \text{global temperature dynamics} \\
& \quad \text{model of the economy} \\
& \quad \text{couplings} + \text{ constraints}
\end{align*}
\]

Objective with stochastic discount rate

Deterministic dynamics and constraints

Observations

- Stoch. discount rate $\rightarrow$ stochastic objective, deterministic states
- Stoch. system parameters $\rightarrow$ stochastic objective, stochastic states
Uncertainty Quantification – sampling

Sampling-based methods:

\[ 0 = M(P, X), \]
\[ P \text{ with } \rho_P(\xi) \]
\[ X = M^{-1}(P), \]
\[ X \text{ with } \rho_X(\xi) \]

→ Successful application in many (engineering) disciplines
→ Yet sampling is usually computationally expensive
→ Here: Polynomial Chaos Expansions [Wiener 1938, Xiu 2010, ....]
Polynomial Chaos Expansion

- Conceptually equivalent to eigenfunctions/Fourier series

\[ P = \mu_P + \sigma_P \Xi, \quad \Xi \sim \mathcal{N}(0, 1) \quad \xrightarrow{?!} \quad P = \alpha + \beta \Xi + \gamma \Xi^2 + \ldots, \quad \alpha, \beta, \gamma \in \mathbb{R} \]

- Random variables described by deterministic PCE coefficients

\[ X = \tilde{X}(\Xi) = \sum_{l=0}^{L} x_l \phi_l(\Xi) \quad \text{with} \quad x_l = \frac{\langle X(\Xi), \phi_l(\Xi) \rangle_{L^2}}{\langle \phi_l(\Xi), \phi_l(\Xi) \rangle_{L^2}} \]

- Efficient sampling

- Extension to multivariate random variables

- **Main idea:**
  Replace stochastic equation by a set of deterministic equations

\[ \rightarrow \text{Applicable to non-Gaussian uncertainties} \]

\[ \rightarrow \text{Not restricted to support being real line} \]
Fourier Series vs. Polynomial Chaos Expansion

<table>
<thead>
<tr>
<th></th>
<th>Fourier Series Expansion</th>
<th>Polynomial Chaos Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>orthogonal basis</td>
<td>${\phi_k(t) = e^{ikt}}_{k=-\infty}^{\infty}$</td>
<td>${\phi_k(\Xi)}_{k=0}^{\infty}$ with deg $\phi_k(\Xi) = k$</td>
</tr>
<tr>
<td>Hilbert space</td>
<td>$\mathcal{H} = L^2(\mathbb{R}/2\pi, \mathbb{C}) \cap C^0$</td>
<td>$L^2(\Omega, \mu; \mathbb{R})$ with $(\Omega, \mathcal{F}, \mu)$</td>
</tr>
<tr>
<td>scalar product</td>
<td>$\langle f, g \rangle_{\mathcal{H}} = \int_{-\pi}^{\pi} f(t)g(t)dt$</td>
<td>$\langle X, Y \rangle_{L^2} = \int_{\Omega} X(\Xi)Y(\Xi)\mu(\Xi)d\mu(\Xi)$</td>
</tr>
<tr>
<td>norm</td>
<td>$|f|<em>{\mathcal{H}} = \sqrt{\langle f, f \rangle</em>{\mathcal{H}}}$</td>
<td>$|X|<em>{L^2} = \left(\int</em>{\Omega} X(\Xi)^2\mu(\Xi)d\mu(\Xi)\right)^{1/2}$</td>
</tr>
<tr>
<td>expansion</td>
<td>$f(t) = \sum_{k=-\infty}^{\infty} f_k \phi_k(t)$</td>
<td>$X(\Xi) = \sum_{l=0}^{\infty} x_l \phi_l(\Xi)$</td>
</tr>
<tr>
<td>truncation</td>
<td>$\tilde{f}(t) = \sum_{k=-n}^{n} f_k \phi_k(t)$</td>
<td>$\tilde{X} = \sum_{l=0}^{L} x_l \phi_l(\Xi)$</td>
</tr>
<tr>
<td>coefficients</td>
<td>$f_k = \frac{\langle f, \phi_k(t) \rangle_{\mathcal{H}}}{\langle \phi_k(t), \phi_k(t) \rangle_{\mathcal{H}}}$</td>
<td>$x_l = \frac{\langle X, \phi_l(\Xi) \rangle_{L^2}}{\langle \phi_l(\Xi), \phi_l(\Xi) \rangle_{L^2}}$</td>
</tr>
<tr>
<td>optimality</td>
<td>$|f - \tilde{f}|<em>{\mathcal{H}} = \min</em>{g \in \mathcal{U}} |f - g|_{\mathcal{H}}$</td>
<td>$|X - \tilde{X}|<em>{L^2} = \min</em>{Y \in \mathcal{U}} |X - Y|_{L^2}$</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{U} = \text{span}{\phi_k(t)}_{k=-n}^{n}$</td>
<td>$\mathcal{U} = \text{span}{\phi_l(\Xi)}_{l=0}^{L}$</td>
</tr>
</tbody>
</table>

PCE $\approx$ Fourier series for random variables
Askey-scheme

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Support</th>
<th>Polynomial Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beta</td>
<td>(0, 1)/[0, 1]</td>
<td>Jacobi</td>
</tr>
<tr>
<td>Gamma</td>
<td>(0, ∞)</td>
<td>Laguerre</td>
</tr>
<tr>
<td>Gaussian</td>
<td>(−∞, ∞)</td>
<td>Hermite</td>
</tr>
<tr>
<td>Uniform</td>
<td>[−1, 1]</td>
<td>Legendre</td>
</tr>
</tbody>
</table>

- Applicable to non-Gaussian uncertainties
- Not restricted to support being real line

How to apply to DICE?
Back to uncertain discount rates in DICE

\[
\begin{align*}
\text{maximize} & \quad W = \sum_{k=0}^{N} (1 + \rho)^{-k} \cdot U(k) \\
\text{subject to} & \quad \text{dynamics of global carbon cycle} \\
& \quad \text{global temperature dynamics} \\
& \quad \text{model of the economy} \\
& \quad \text{couplings} + \text{constraints}
\end{align*}
\]

\[
\begin{align*}
\text{maximize} & \quad E[W] - 0.1 \cdot Var[W] = \sum_{k=0}^{N} \tilde{\alpha}(k) \cdot U(k) - 0.1\Phi(\tilde{\alpha}(k), U(k)) \\
E[W] & = \sum_{k=0}^{N} \tilde{\alpha}(k) \cdot U(k), \quad \tilde{\alpha}(k) \text{ pre-computed via PCE} \\
Var[W] & = \sum_{k=0}^{N} \Phi(\tilde{\alpha}(k), U(k)) , \quad \tilde{\alpha}(k) \text{ pre-computed via PCE, } \Phi \text{ polynomial}
\end{align*}
\]
Model for stochastic discount rate

- Beta distribution with $PDF(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1 - x)^{\beta-1}$

- Stochastic discount rate $\rho \in [\underline{\rho}, \bar{\rho}]$ mapped to compact support $x \in [0, 1]$

- Different shapes of $PDF$ can be matched via $\alpha$ and $\beta$
Model for stochastic discount rate (cont’d)

- Beta distribution with \( PDF(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1 - x)^{\beta-1} \)

- Stochastic discount rate \( \rho \in [\underline{\rho}, \overline{\rho}] \) mapped to compact support \( x \in [0, 1] \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \rho )</th>
<th>( \overline{\rho} )</th>
<th>( E[\rho] )</th>
<th>( 10^4 \cdot Var[\rho] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0.032</td>
<td>0.015</td>
<td>0.015</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>0</td>
<td>0.070</td>
<td>0.035</td>
<td>0.140</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>0</td>
<td>0.055</td>
<td>0.039</td>
<td>0.037</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0</td>
<td>0.055</td>
<td>0.015</td>
<td>0.037</td>
</tr>
</tbody>
</table>

- Considered scenarios:

![Considered Beta distributions](image-url)
Back to uncertain discount rates in DICE (cont’d)

\[ E[W] = \sum_{k=0}^{N} \bar{\alpha}(k) \cdot U(k), \quad \bar{\alpha}(k) \text{ pre-computed via PCE} \]

\[ Var[W] = \sum_{k=0}^{N} \Phi(\bar{\alpha}(k), U(k)), \quad \bar{\alpha}(k) \text{ pre-computed via PCE, } \Phi \text{ polynomial} \]

Normalized objective \( W_1 : U(k) = 1, \quad \forall k \)

Polynomial chaos provides reasonably accurate approximation.
Results without temperature target

\[ SCC = \frac{dC}{dE} \]

\[ P(\rho)/C \]

\[ \text{Emissions Control Rate} \]

\[ \text{Savings Rate} \]
Results without temperature target (cont’d)
Results with temperature target

**SCC = dC/dE**

Considered Beta distributions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Density</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 2, \beta = 2$</td>
<td>Blue</td>
</tr>
<tr>
<td>$\alpha = 2, \beta = 2$</td>
<td>Green</td>
</tr>
<tr>
<td>$\alpha = 2, \beta = 5$</td>
<td>Red</td>
</tr>
<tr>
<td>$\alpha = 5, \beta = 2$</td>
<td>Black</td>
</tr>
</tbody>
</table>

**Emissions Control Rate**

**Savings Rate**
Results with temperature target (cont’d)
Intermediate summary

- Stochastic discounting leads to reduced Social Cost of Carbon (in the long run)
- With temperature target → differences between different distributions

- Mathematical explanation?
- Receding horizon variant?
Insights into stochastic discounting

\[ W = \sum_{k=0}^{N} \alpha(k) \cdot U(k), \quad \alpha(k) = (1 + \rho)^{-k} \]

\[ E[W] = \sum_{k=0}^{N} \tilde{\alpha}(k) \cdot U(k), \quad \tilde{\alpha}(k) \text{ pre-computed via PCE} \]

\[ Var[W] = \sum_{k=0}^{N} \Phi(\tilde{\alpha}(k), U(k)), \quad \tilde{\alpha}(k) \text{ pre-computed via PCE, } \Phi \text{ polynomial} \]

Normalized objective \( W_1 : U(k) = 1, \ \forall k \)
Insights into stochastic discounting

\[ W = \sum_{k=0}^{N} \alpha(k) \cdot U(k), \quad \alpha(k) = (1 + \rho)^{-k} \]

\[ E[W] = \sum_{k=0}^{N} \bar{\alpha}(k) \cdot U(k), \quad \bar{\alpha}(k) \text{ pre-computed via PCE} \]

\[ Var[W] = \sum_{k=0}^{N} \Phi(\bar{\alpha}(k), U(k)), \quad \bar{\alpha}(k) \text{ pre-computed via PCE, } \Phi \text{ polynomial} \]

Normalized objective \( W_1 : U(k) = 1, \quad \forall k \)

Mean of stochastic discounting is equivalent to non-exponential det. discounting!
Summary

- DICE 2013 with decision making under uncertainty
- Stochastic discounting (Beta distributions)
- Stochastic discounting = non-exponential discounting
- Efficient numerics available

Outlook

- Other distributions? Relation to loss aversion?
- Relation to time-varying discounting (DDR)?
- MPC? Stochastic system parameters? ...

Acknowledgements

- Frederick Zahn (KIT)
- Funding: Daimler und Benz Stiftung

Thanks! Questions?