# Towards Climate-Economy Assessment with Stochastic Discount Rates

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# Complete DICE model

$$\max_{\mu(k),s(k)} \; W = \sum_{k=0}^{N} (1+\rho)^{-k} U(k)$$

subject to, for all $k = 0, \ldots, N-1$

$$\begin{bmatrix} M(k+1) \\ T(k+1) \\ K(k+1) \end{bmatrix} = \begin{bmatrix} \Phi & 0 & 0 \\ 0 & \Psi & 0 \\ 0 & 0 & (1-\delta)^5 \end{bmatrix} \begin{bmatrix} M(k) \\ T(k) \\ K(k) \end{bmatrix} + 5 \begin{bmatrix} B_M E(K(k),\, \mu(k),\, d(k)) \\ B_T F(M_{AT}(k),\, d(k)) \\ f(K(k),\, T_{AT}(k),\, s(k),\, \mu(k),\, d(k)) \end{bmatrix}$$

$$\left[M(0),\, T(0),\, K(0)\right]^T = \left[M_0,\, T_0,\, K_0\right]^T$$

$$0 \leq \left[M(k),\, T(k),\, K(k)\right]^T$$

$$0 \leq \mu(k) \leq 1$$

$$0 \leq s(k) \leq 1$$

$$2 \geq T_{AT}(k)$$

How to account for uncertainties in DICE?

- Feedback policy
- Robustified or stochastic computations

# Climate-economy assessment via optimization

$$\underset{\text{savings, } CO_2 \text{ mitigation}}{\text{maximize}} \quad W = \sum_{k=0}^{N} (1+\rho)^{-k} \cdot U(k)$$

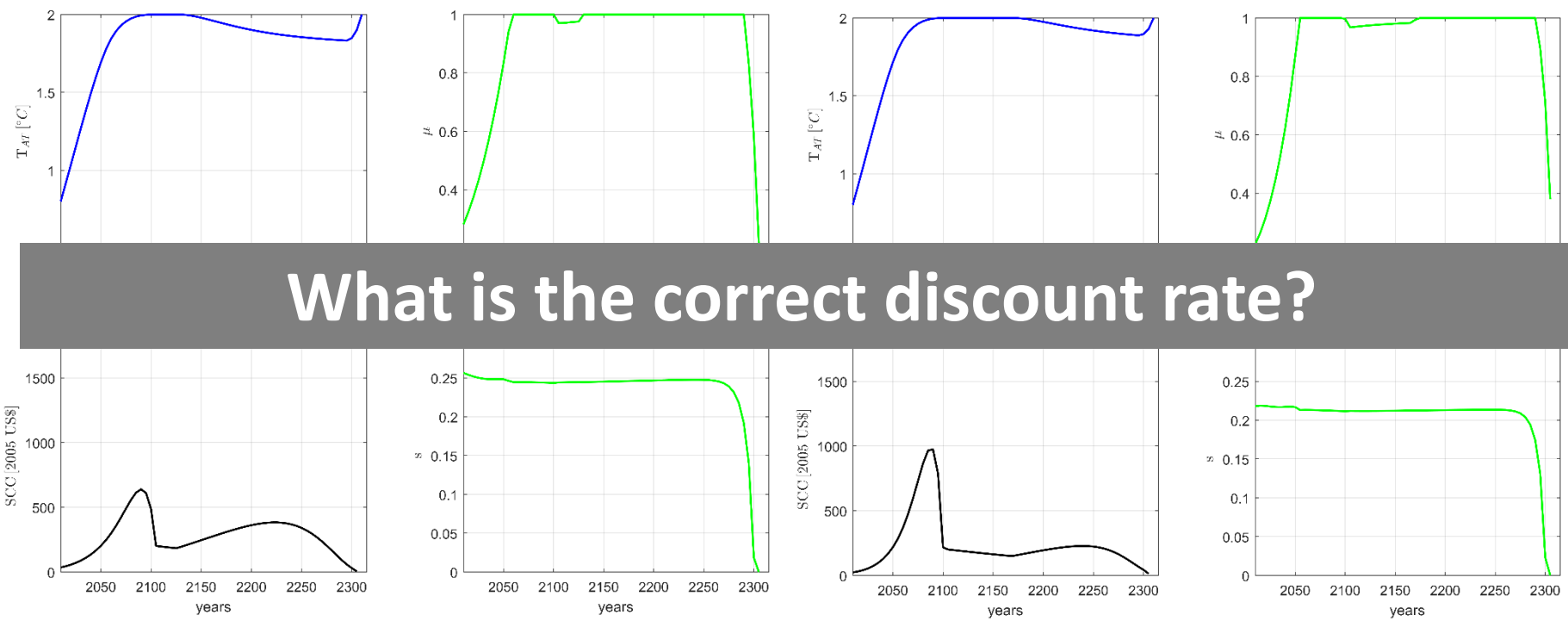
subject to dynamics of global carbon cycle

global temperature dynamics

model of the economy

couplings + constraints

Discount rate $\rho = 0.015$ Discount rate $\rho = 0.03$




**What is the correct discount rate?**

# Outline

Discount rates in DICE?

Computing the Social Cost of Carbon

How to do computations with stochastic uncertainties?

First results for Polynomial Chaos Expansions in DICE 2013

Summary & outlook

# Social Cost of Carbon (SCC)

[The] *social cost of carbon in a particular year is the decrease in aggregate consumption in that year that would change the current* […] *value of social welfare by the same amount as a one unit increase in carbon emissions* [1t$CO_2$] *in that year.*

S. C. Newbold, C. Griffiths, C. Moore, A. Wolverton, and E. Kopits. *A rapid assessment model for understanding the social cost of carbon.* Clim. Change Econ., 2013

$$SCC(k) = -\frac{\text{marginal impact of emissions on welfare}}{\text{marginal impact of consumption on welfare}} = -\frac{\frac{\partial W}{\partial E(k)}}{\frac{\partial W}{\partial C(k)}}$$

# How to compute the SCC?

Code the NLP

$$\max_{\mu(k),s(k)} W = \sum_{k=0}^{N} (1+\rho)^{-k} U(k)$$

subject to dynamics of global carbon cycle

global temperature dynamics

model of the economy

couplings $+$ constraints

such that the equality constraints

$$E(k) = \dots$$
$$C(k) = \dots$$

appear explicitly

$\Rightarrow$ $\frac{\partial W}{\partial E(k)}$ and $\frac{\partial W}{\partial C(k)}$ = corresponding Lagrange multipliers

C. Kellett, T. Faulwasser and S. Weller. DICE2013R-mc: A Matlab / CasADi Implementation of Vanilla DICE2013R github.com 2016

# Uncertain discount rates in DICE

$$\underset{\text{savings, } CO_2 \text{ mitigation}}{\text{maximize}} \quad W = \sum_{k=0}^{N} (1+\rho)^{-k} \cdot U(k)$$

Objective with stochastic discount rate

subject to
- dynamics of global carbon cycle
- global temperature dynamics
- model of the economy
- couplings + constraints

Deterministic dynamics and constraints

**Observations**

- Stoch. discount rate $\rightarrow$ stochastic objective, deterministic states
- Stoch. system parameters $\rightarrow$ stochastic objective, stochastic states

# Uncertainty Quantification – sampling

Sampling-based methods:

$0 = M(\mathsf{P}, \mathsf{X}),$

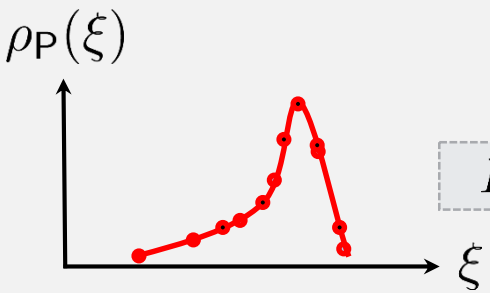
P with $\rho_{\mathsf{P}}(\xi)$

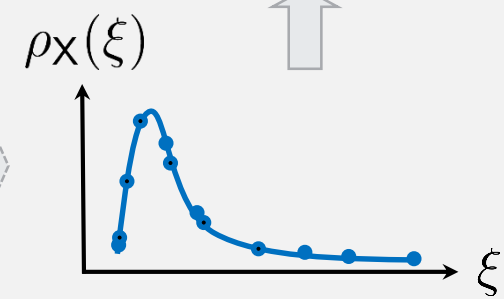
difficult

$\mathsf{X} = M^{-1}(\mathsf{P}),$

X with $\rho_{\mathsf{X}}(\xi)$

$K$ deterministic problems




→ Successful application in many (engineering) disciplines

→ *Yet sampling is usually computationally expensive*

→ *Here: **Polynomial Chaos Expansions** [Wiener 1938, Xiu 2010, ….]*

# Polynomial Chaos Expansion

- Conceptually equivalent to eigenfunctions/Fourier series

$$\mathsf{P} = \mu_{\mathsf{P}} + \sigma_{\mathsf{P}}\Xi,\ \Xi \sim \mathcal{N}(0,1) \overset{?!}{\longleftrightarrow} \mathsf{P} = \alpha + \beta\Xi + \gamma\Xi^2 + \ldots,\ \alpha, \beta, \gamma \in \mathbb{R}$$

- Random variables described by deterministic PCE coefficients

$$\mathsf{X} = \tilde{X}(\Xi) = \sum_{l=0}^{L} x_l \phi_l(\Xi) \quad \text{with} \quad x_l = \frac{\langle X(\Xi), \phi_l(\Xi)\rangle_{\mathsf{L}^2}}{\langle \phi_l(\Xi), \phi_l(\Xi)\rangle_{\mathsf{L}^2}}$$

- Efficient sampling
- Extension to multivariate random variables
- **Main idea:**
  Replace stochastic equation by a set of deterministic equations

→ Applicable to non-Gaussian uncertainties

→ Not restricted to support being real line

# Fourier Series vs. Polynomial Chaos Expansion

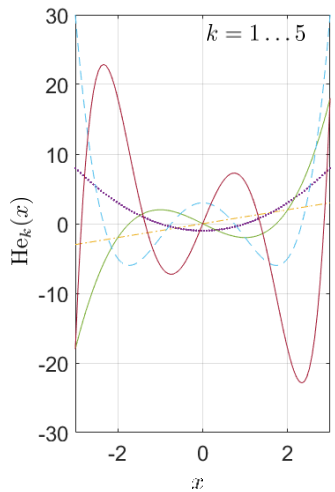
| | Fourier Series Expansion | Polynomial Chaos Expansion |
|---|---|---|
| orthogonal basis | $\{\phi_k(t) = \mathrm{e}^{\mathrm{i}kt}\}_{k=-\infty}^{\infty}$ | $\{\phi_k(\Xi)\}_{k=0}^{\infty}$ with $\deg \phi_k(\Xi) = k$ |
| Hilbert space | $\mathcal{H} = \mathcal{L}^2(\mathbb{R}/2\pi, \mathbb{C}) \cap C^0$ | $\mathsf{L}^2(\Omega, \mu; \mathbb{R})$ with $(\Omega, \mathcal{F}, \mu)$ |
| scalar product | $\langle f, g\rangle_{\mathcal{H}} = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\mathrm{d}t$ | $\langle \mathsf{X}, \mathsf{Y}\rangle_{\mathsf{L}^2} = \mathrm{c}$ |
| norm | $\lVert f\rVert_{\mathcal{H}} = \sqrt{\langle f, f\rangle_{\mathcal{H}}}$ | $\lVert \mathsf{X}$ $\langle \mathsf{X}, \mathsf{X}\rangle_{\mathsf{L}^2}}$ |
| expansion | $f(t) = \sum_{k=-\infty}^{\infty} f_k \phi_k(t)$ | $= X(\Xi) = \sum_{l=0}^{\infty} x_l \phi_l(\Xi)$ |
| truncation | $\tilde{f}(t) = \sum^{n}$ | $\tilde{\mathsf{X}} = \tilde{X}(\Xi) = \sum_{l=0}^{L} x_l \phi_l(\Xi)$ |
| coefficients | $\phi_l(t)\rangle_{\mathcal{H}}$ / $\langle \phi_l(t), \phi_l(t)\rangle_{\mathcal{H}}$ | $x_l = \dfrac{\langle \mathsf{X}, \phi_l(\Xi)\rangle_{\mathsf{L}^2}}{\langle \phi_l(\Xi), \phi_l(\Xi)\rangle_{\mathsf{L}^2}}$ |
| optimality | $\lVert f - \tilde{f}\rVert_{\mathcal{H}} = \min_{g \in \mathcal{U}} \lVert f - g\rVert_{\mathcal{H}}$ | $\lVert \mathsf{X} - \tilde{\mathsf{X}}\rVert_{\mathsf{L}^2} = \min_{\mathsf{Y} \in \mathfrak{U}} \lVert \mathsf{X} - \mathsf{Y}\rVert_{\mathsf{L}^2}$ |
| | $\mathcal{U} = \mathrm{span}\{\phi_k(t)\}_{k=-n}^{n}$ | $\mathfrak{U} = \mathrm{span}\{\phi_l(\Xi)\}_{l=0}^{L}$ |

PCE ≈ Fourier series for random variables

# Polynomial Chaos Expansion – Bases

→ Askey-scheme

| Distribution | Support | Polynomial Basis |
|---|---|---|
| Beta | $(0,1)/[0,1]$ | Jacobi |
| Gamma | $(0,\infty)$ | Laguerre |
| Gaussian | $(-\infty,\infty)$ | Hermite |
| Uniform | $[-1,1]$ | Legendre |




→ Applicable to non-Gaussian uncertainties

→ Not restricted to support being real line

How to apply to DICE?

# Back to uncertain discount rates in DICE

$$\underset{\text{savings, } CO_2 \text{ mitigation}}{\text{maximize}} \quad W = \sum_{k=0}^{N} (1+\rho)^{-k} \cdot U(k)$$

subject to dynamics of global carbon cycle

global temperature dynamics

model of the economy

couplings + constraints

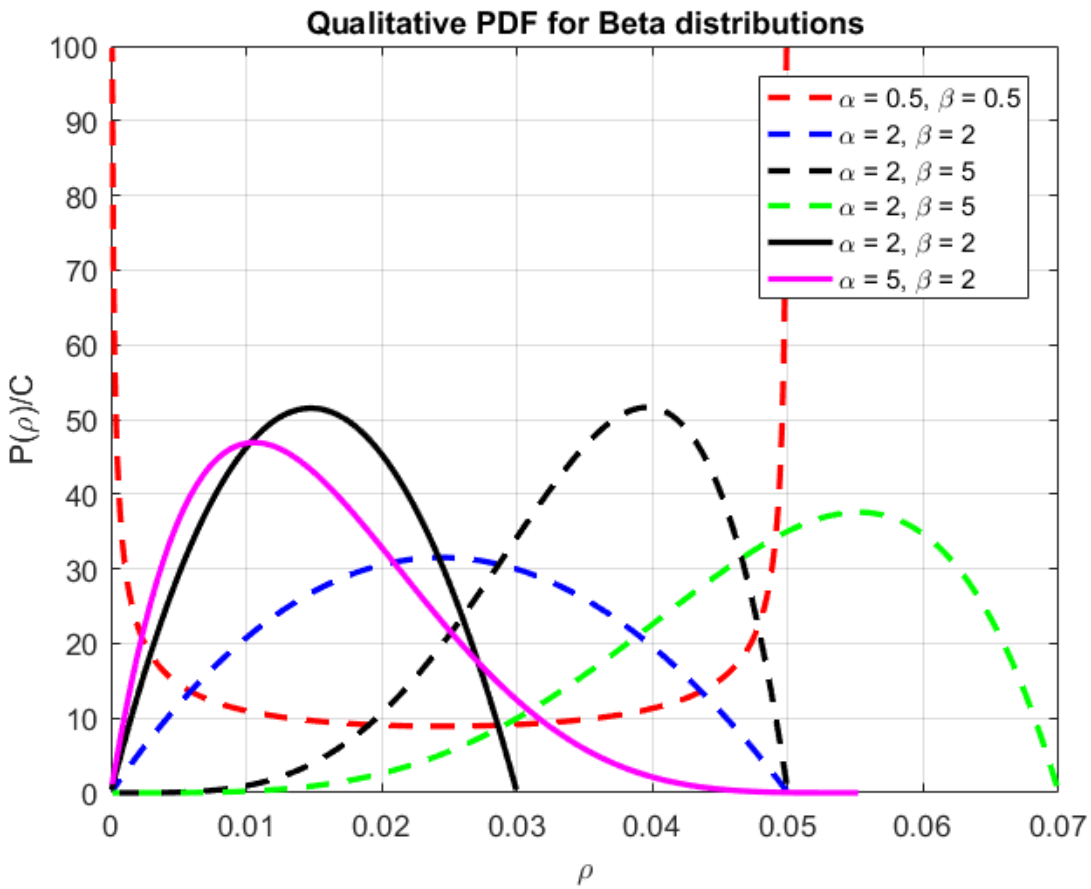
$$\underset{\text{savings, } CO_2 \text{ mitigation}}{\text{maximize}} \quad E[W] - 0.1 \cdot Var[W] = \sum_{k=0}^{N} \bar{\alpha}(k) \cdot U(k) - 0.1\Phi(\tilde{\alpha}(k), U(k))$$

$$E[W] = \sum_{k=0}^{N} \bar{\alpha}(k) \cdot U(k), \quad \bar{\alpha}(k) \text{ pre-computed via PCE}$$

$$Var[W] = \sum_{k=0}^{N} \Phi(\tilde{\alpha}(k), U(k)), \quad \tilde{\alpha}(k) \text{ pre-computed via PCE, } \Phi \text{ polynomial}$$

# Model for stochastic discount rate

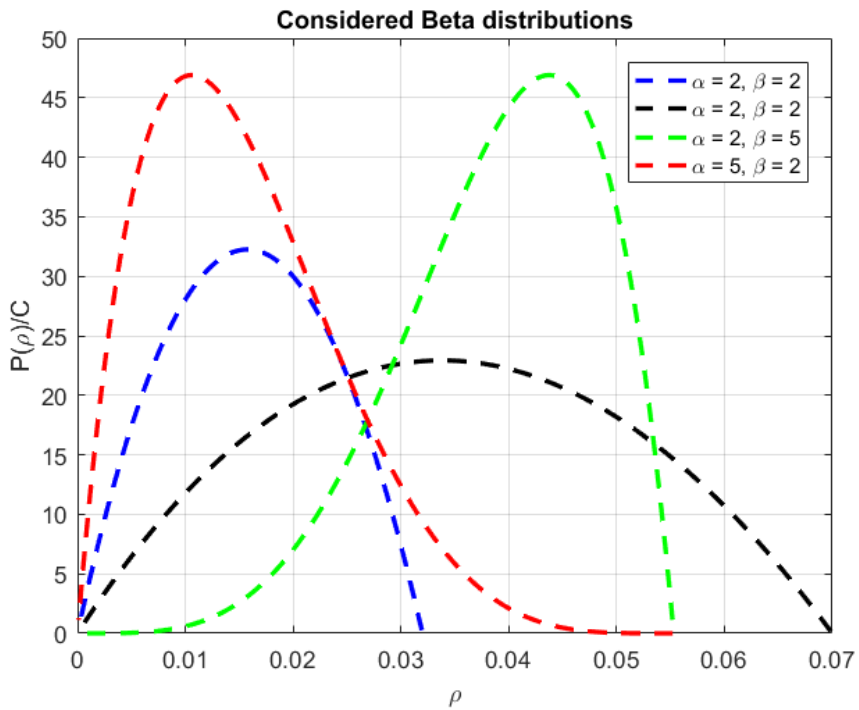
- Beta distribution with $PDF(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$
- Stochastic discount rate $\rho \in [\underline{\rho}, \overline{\rho}]$ mapped to compact support $x \in [0, 1]$
- Different shapes of $PDF$ can be matched via $\alpha$ and $\beta$



# Model for stochastic discount rate (cont'd)

- Beta distribution with $PDF(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$
- Stochastic discount rate $\rho \in [\underline{\rho}, \overline{\rho}]$ mapped to compact support $x \in [0, 1]$
- Considered scenarios:

| $\alpha$ | $\beta$ | $\underline{\rho}$ | $\overline{\rho}$ | $E[\rho]$ | $10^4 \cdot Var[\rho]$ |
|---|---|---|---|---|---|
| 2 | 2 | 0 | 0.032 | 0.015 | 0.015 |
| 2 | 2 | 0 | 0.070 | 0.035 | 0.140 |
| 2 | 5 | 0 | 0.055 | 0.039 | 0.037 |
| 5 | 2 | 0 | 0.055 | 0.015 | 0.037 |

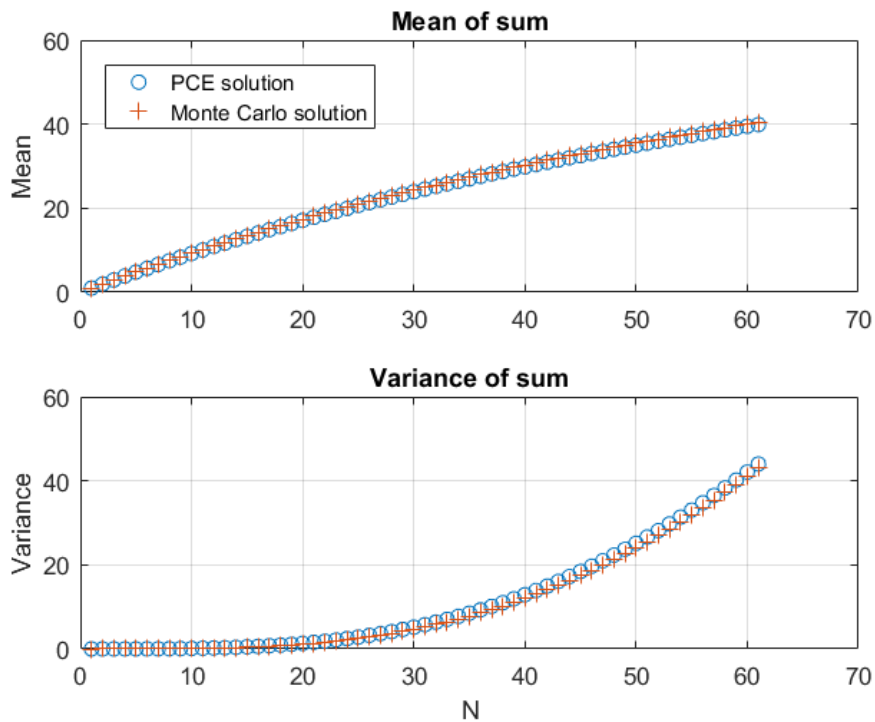


# Back to uncertain discount rates in DICE (cont'd)

$$E[W] = \sum_{k=0}^{N} \bar{\alpha}(k) \cdot U(k), \quad \bar{\alpha}(k) \text{ pre-computed via PCE}$$

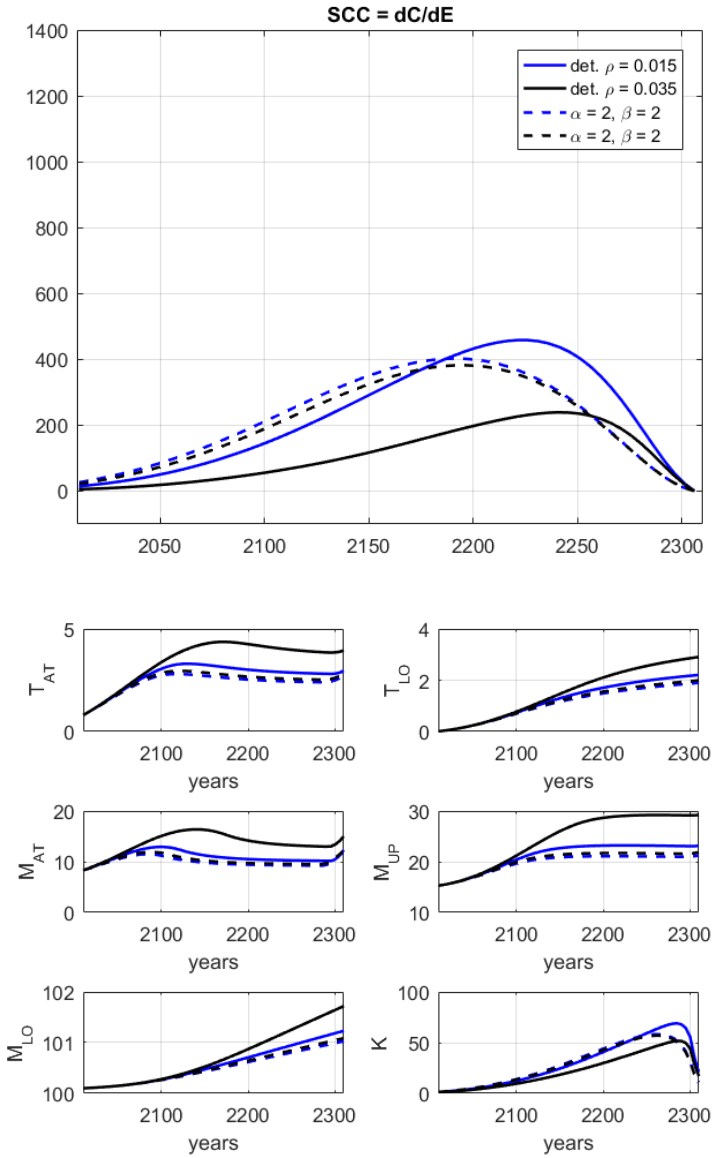
$$Var[W] = \sum_{k=0}^{N} \Phi\left(\tilde{\alpha}(k), U(k)\right), \quad \tilde{\alpha}(k) \text{ pre-computed via PCE, } \Phi \text{ polynomial}$$

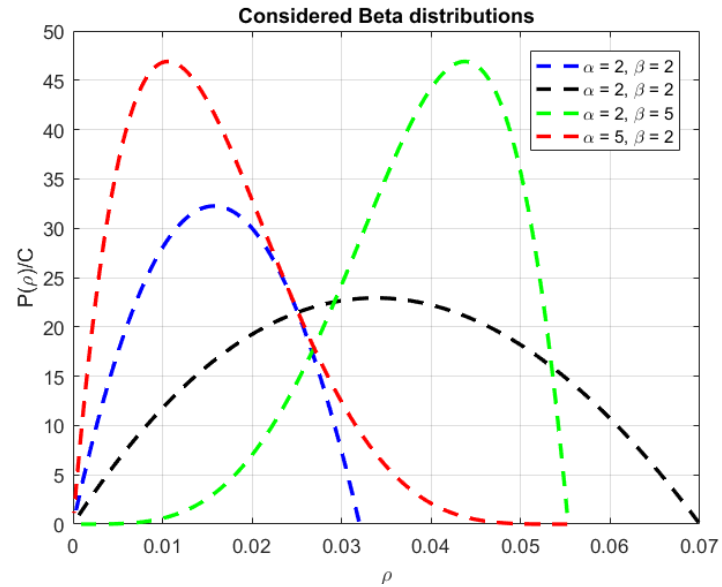
Normalized objective $W_1 : U(k) = 1, \quad \forall k$

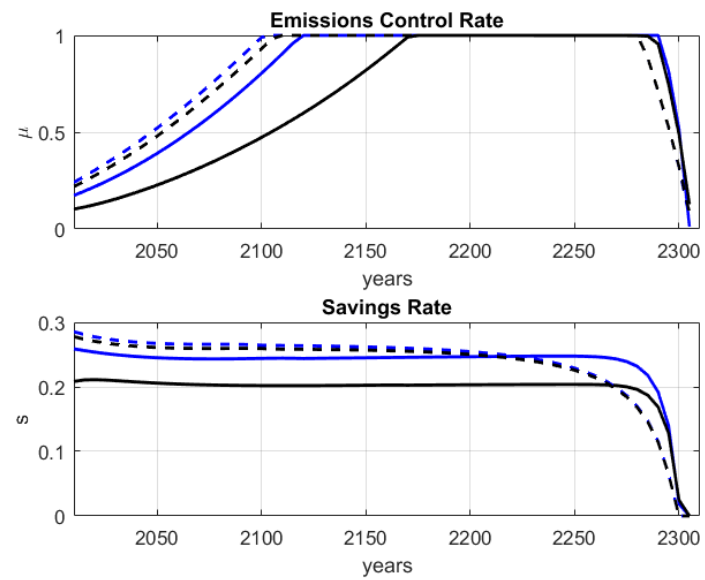



Polynomial chaos provides reasonably accurate approximation.

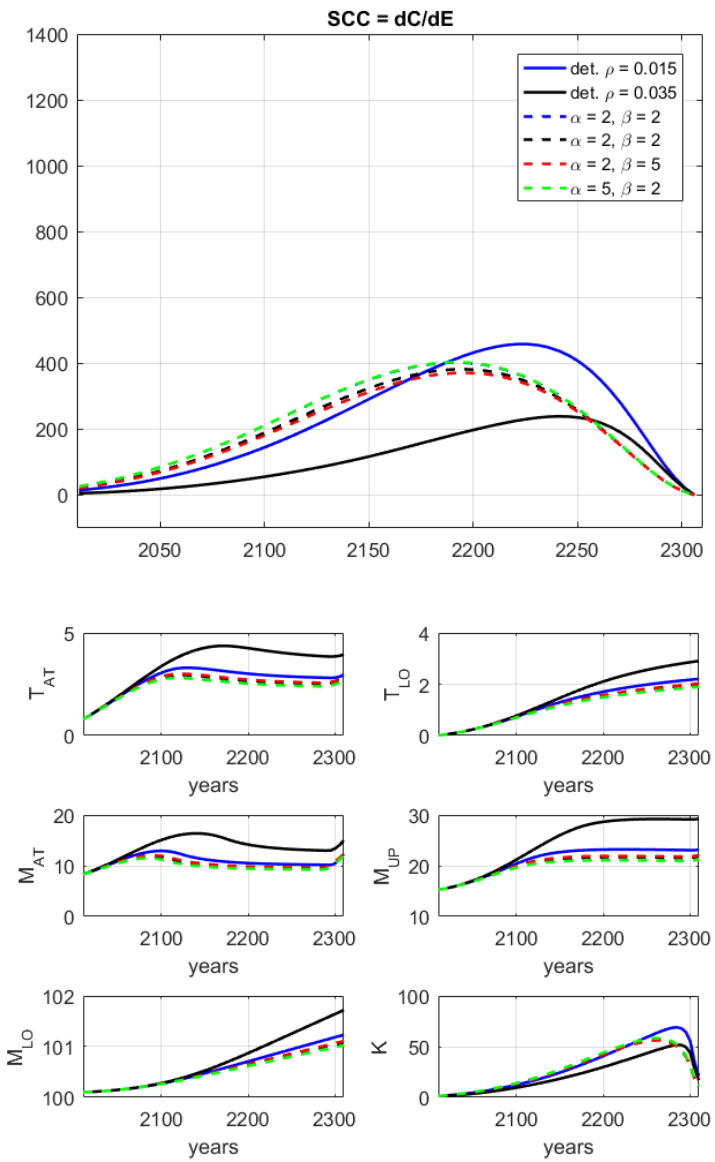
# Results without temperature target

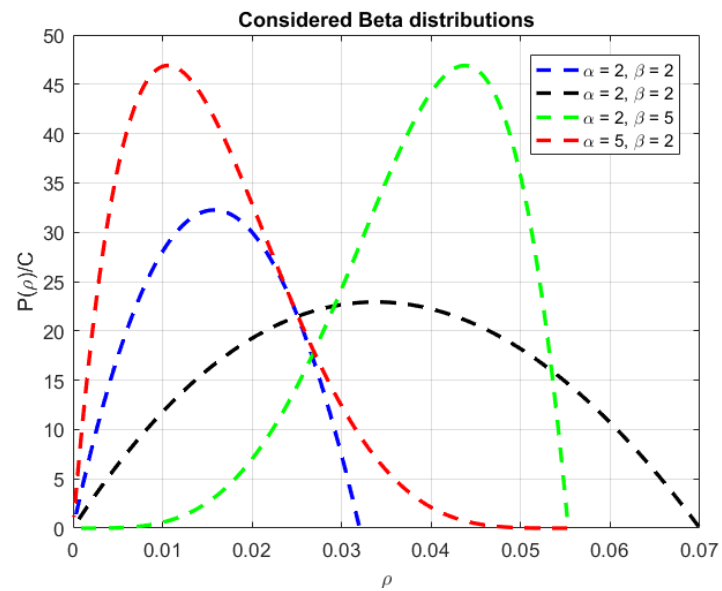


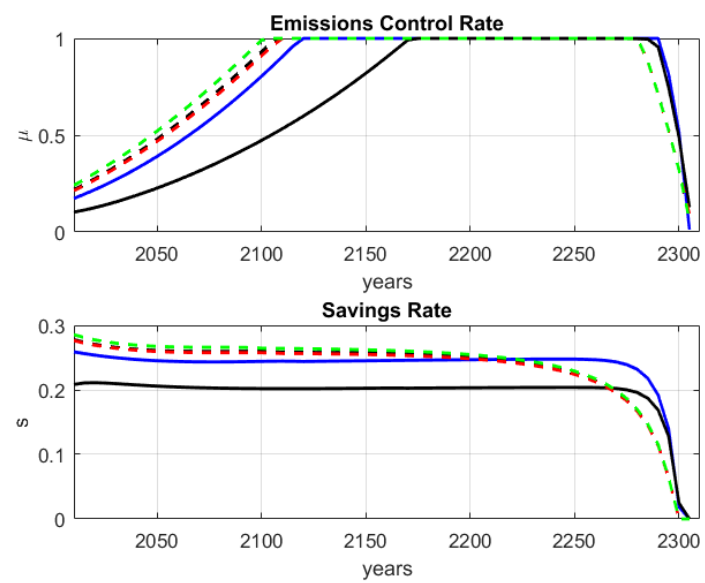



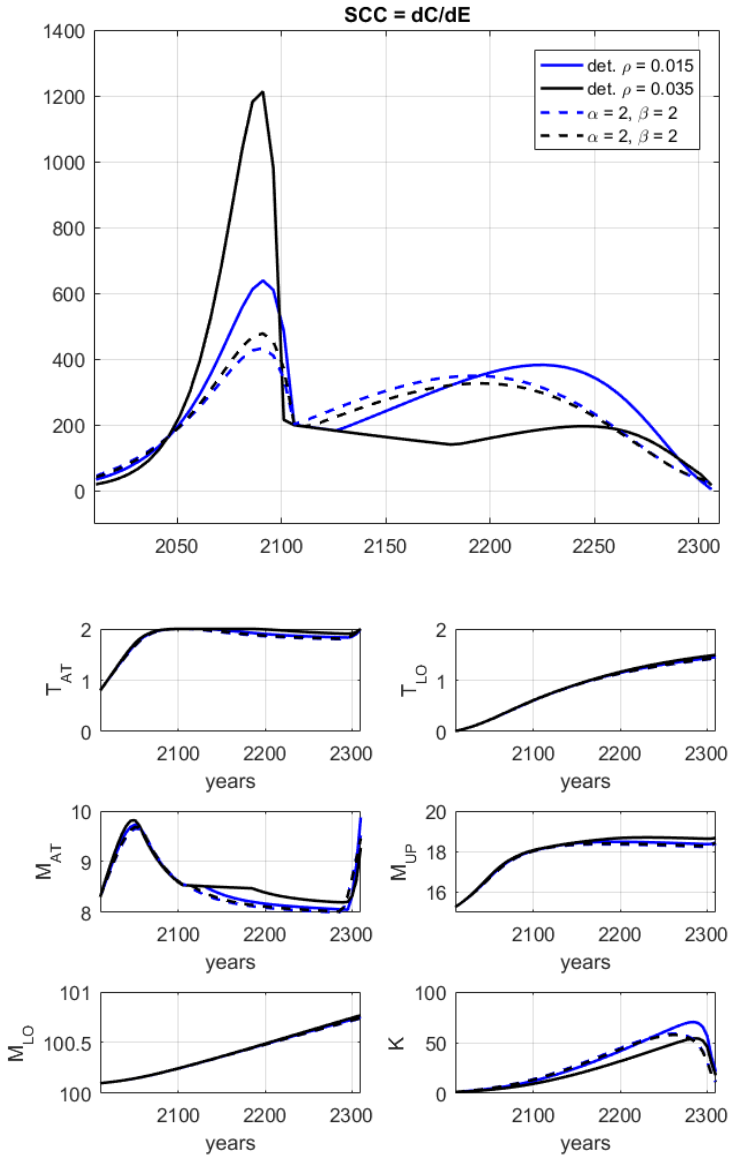
# Results without temperature target (cont'd)

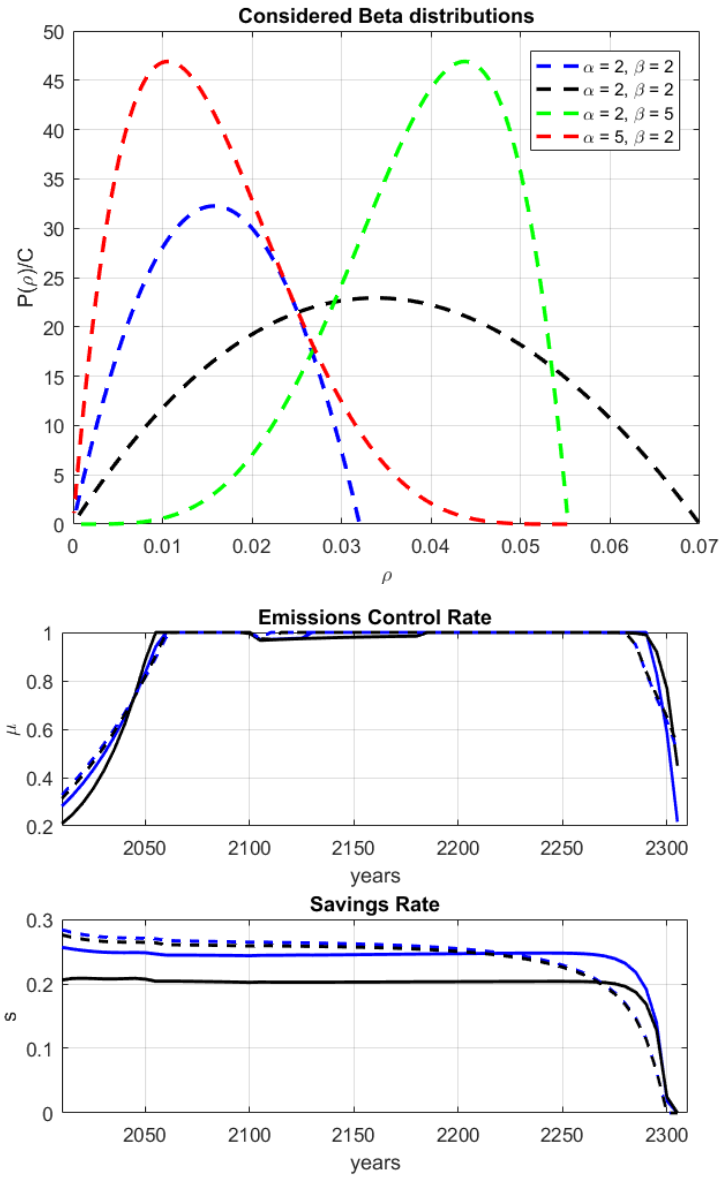


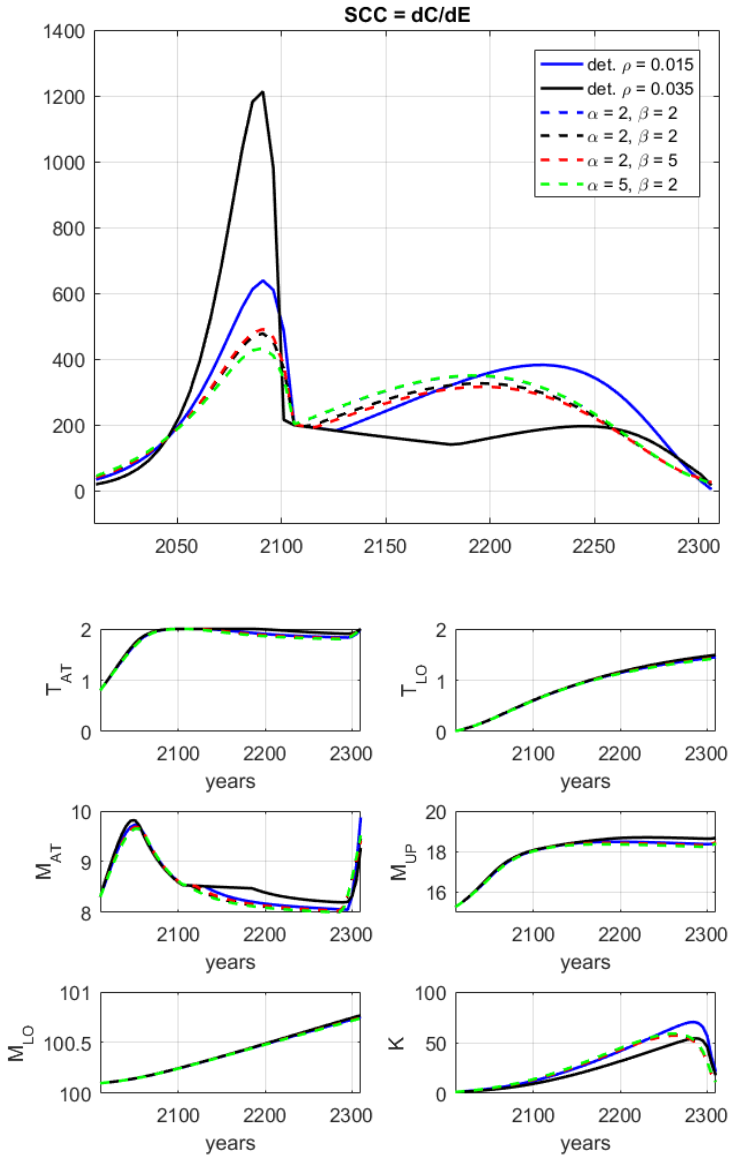



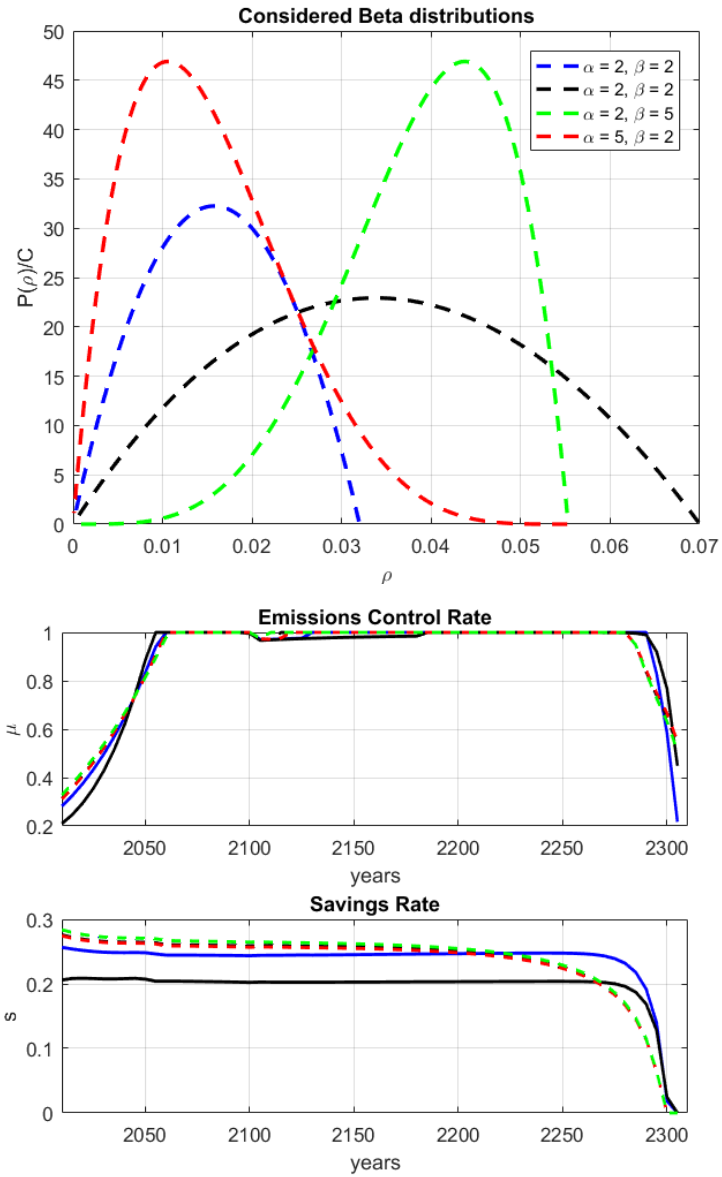
# Results with temperature target



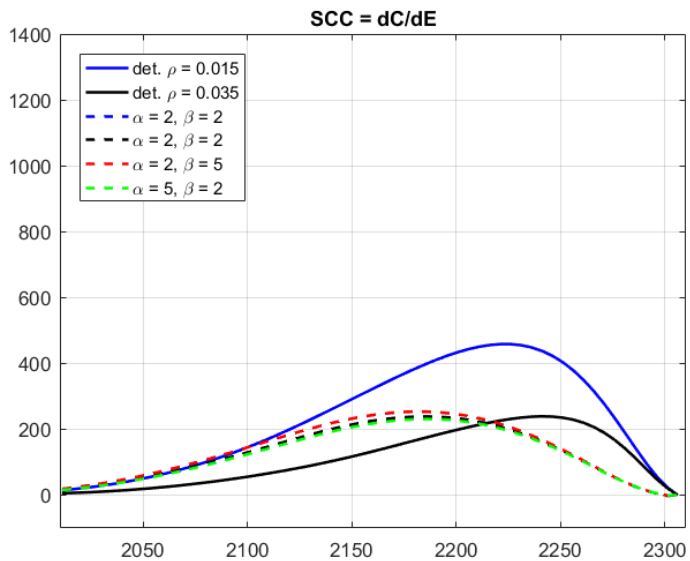



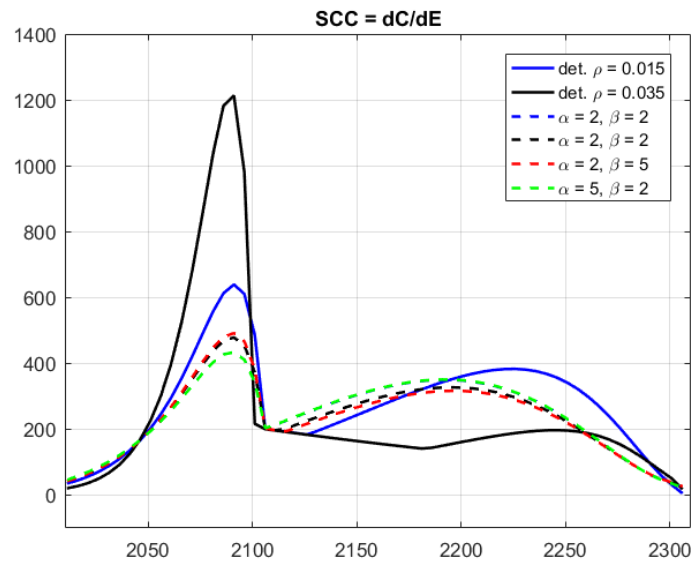
# Results with temperature target (cont'd)



# Intermediate summary




- Stochastic discounting leads to reduced Social Cost of Carbon (in the long run)
- With temperature target → differences between different distributions

- Mathematical explanation?
- Receding horizon variant?

# Insights into stochastic discounting

$$W = \sum_{k=0}^{N} \alpha(k) \cdot U(k), \quad \alpha(k) = (1+\rho)^{-k}$$

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$$Var[W] = \sum_{k=0}^{N} \Phi\left(\tilde{\alpha}(k), U(k)\right), \quad \tilde{\alpha}(k) \text{ pre-computed via PCE}, \Phi \text{ polynomial}$$

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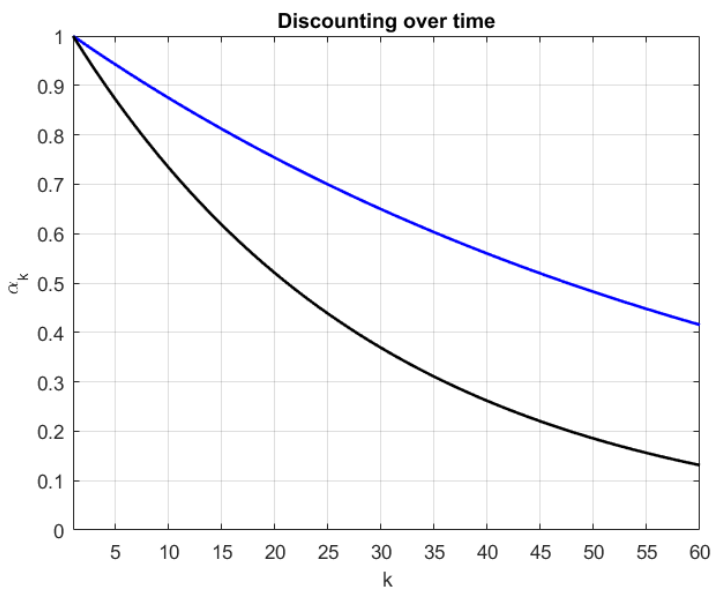


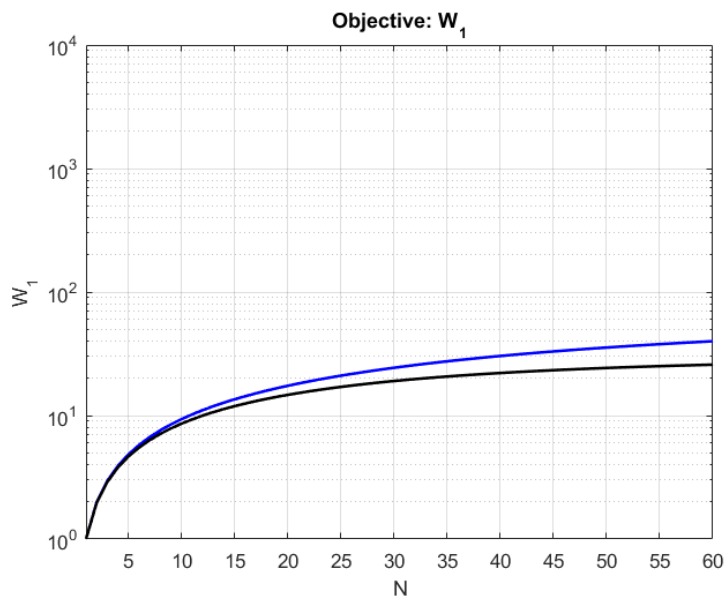

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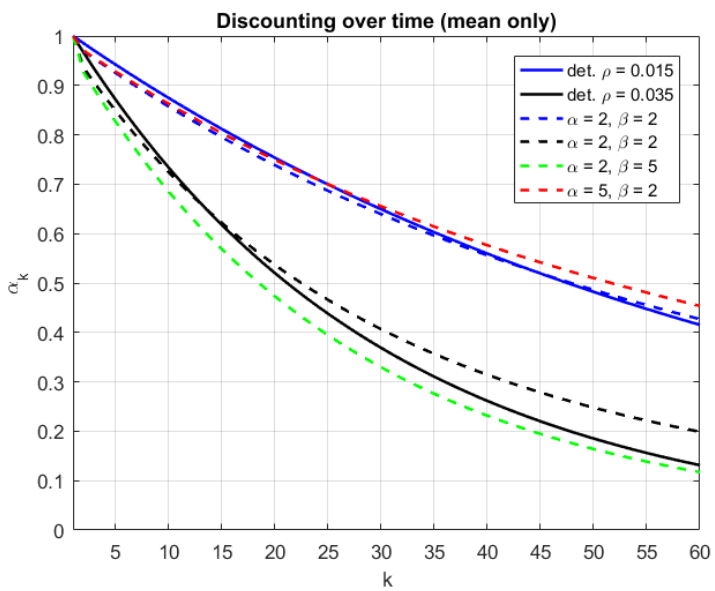


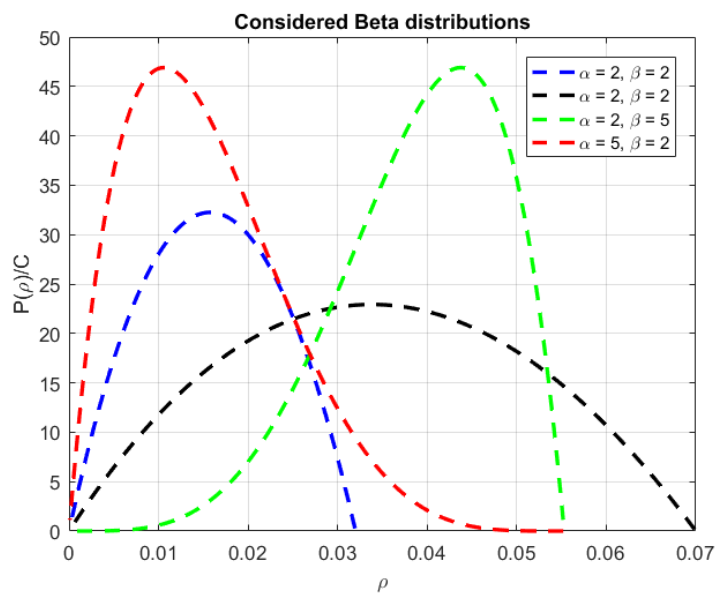


Mean of stochastic discounting is equivalent to non-exponential det. discounting!

# Summary and outlook

## Summary

- DICE 2013 with decision making under uncertainty
- Stochastic discounting (Beta distributions)
- Stochastic discounting = non-exponential discounting
- Efficient numerics available

## Outlook

- Other distributions? Relation to loss aversion?
- Relation to time-varying discounting (DDR)?
- MPC? Stochastic system parameters? …


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**Thanks! Questions?**