Imprecise Probability Bridge Scenario-Forecast Gap

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Abstract

There is a gap in futures research between forecasts, which are assigned precise probabilities, and scenarios which are not. I propose to bridge it using imprecise probabilities: Instead of believing in a single probability distribution, beliefs are represented by a closed convex set of probability distributions, all held equally credible. This allows to define an upper bound for the probability of any given future, a kind of degree of possibility. I then propose three formal rules for scenario analysis: No storyline should be more probable than another, the set of scenarios should be maximally plausible, and it should bracket the largest portion of the possible futures. These three rules outline a mathematical Technique that could play for the arts of the Long View the same role that Linear Perspective played for the arts of Drawing and Painting in the European renaissance. For illustration, these tools are used to conduct a mock analysis about the level of global warming by 2100.

Keywords: Decision-making under uncertainty.

1 Introduction

Both *Scenarios* and *Forecasts* are detailed descriptions of a system's future, with the conventional difference that a set of scenarios is presented without quantifying any degree of confidence or likelihood, while forecasts are attached to a probability distribution. This distinction is a source of permanent tension in Futures study. Decision-makers often demand quantification of uncertainty, but consultants in strategic scenario planning are reluctant to probabilize. Adam Kahane from Shell, as reported in Best [1991], explicits well the problem with probabilities:

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> We don't assign probabilities to our scenarios, for several reasons. First, we intentionally write several scenarios that are more or less equally plausible, so that none is dismissed out of hand. Second, by definition, any given scenario has only an infinitesimal probability of being right because so many variations are possible. Third, the reason to be hesitant about all scenario quantification — not probabilities, economic growth rate or whatever — is that there is a very strong tendency for people to grab onto the numbers and ignore the more important conceptual or structural messages.

In the same discussion, the well-known analyst Peter Wack concluded more sanguinely about probabilities:

> But I have a strong feeling that it will be poisonous and will contaminate the logic of scenarios.

To which David Kline, supervisor of the gas forecasting and model development unit of the California Energy Commission, respectfully disagreed and added:

> Trying to bridge just this kind of gap [...] represents one of the most important intellectual and practical endeavors of our time.

This paper proposes to bridge this gap. The first part reminds a few elements of the mathematical theory of imprecise probabilities [Walley, 1991]. The second discusses three formal rules for selecting scenarios. The third part illustrates the proposed methods by presenting results of an hypothetical analysis about possible global warming futures.

2 Background on imprecise probabilities

2.1 Information with sets of probabilities

A scenario is a point ω in the space of all possible futures Ω. A scenario set S is a subset of Ω. The subset S typically contains 3 or 4 elements, while Ω is typically infinite. Let p denote a probability defined on Ω. As Kahane outlined, the probability $p(S)$ is usually infinitesimal.

But as Knight [1921], Keynes [1921] pointed out long ago, information about the future of socio-economic systems can hardly be represented by a single probability distribution p. By nature scenarios are called for in situations of deep epistemic uncertainty, where the frequency-based justification for a probability distribution is not even weak but irrelevant.

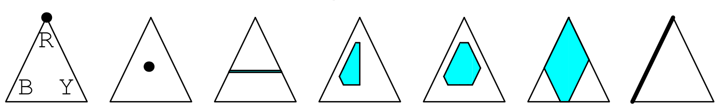
Several lines of research synthesized at `http://ippserv.rug.ac.be` converge to suggest to represent information with a set of probability distributions considering that any p in $\mathcal{C}$ is equally possible [Cozman, 1999]. For example, Ellsberg [1961] imagined an urn containing 30 red balls and 60 black and yellow balls, the latter in unknown proportion. Interested in describing information about the color of a single ball drawn from the urn, he wrote:

> Each subject does know enough about the problem to *rule out* a number of possible distributions. [. . .] He *knows* (by the terms of the experiment) that there are red balls in the urn; in fact, he knows that exactly 1/3 of the balls are red. Thus, in his "choice" of a subjective probability distribution over red, yellow, black — if he wanted such an estimate as a basis for decision — he is limited to the set of potential distributions between (1/3, 2/3, 0) and (1/3, 0, 2/3): i.e. to the infinite set $(1/3,\ \lambda,\ 2/3-\lambda)$, $0 \le \lambda \le 2/3$. Lacking any observations on the number of yellow or black balls, he may have little or no information indicating that one of the remaining, infinite set of distributions is more "likely", more worthy of attention than any other.

This set will be denoted $\mathcal{C}$ hereafter, because it is the set of credible probability distributions. We also assume that $\mathcal{C}$ is closed (the border is included in the set), and convex for mathematical simplicity. In addition to Ellsberg objective interpretation, the subjective interpretation is that the decision-maker believing in $\mathcal{C}$ should not accept any gamble those expected payoff value can be negative for any probability $p \in \mathcal{C}$.

2.2 How to define sets of probabilities

To illustrate the nature of $\mathcal{C}$, let us elaborate graphically on Ellsberg's urn, where $\Omega = \{Red, Black, Yellow\}$. Any triple $(p_{Red}, p_{Black}, p_{Yellow})$ can be represented as a point in a triangle those summits correspond to (1,0,0), (0,1,0) and (0,0,1). Geometrically $\mathcal{C}$ can be represented as a shaded area inside that triangle. Putting for example Red on the top summit, Black left and Yellow right, consider this lineup of different $\mathcal{C}$:



The first, leftmost triangle with a single dot at the top summit represents the information that the urn is comprised of all "red" balls: the color of the ball that will be drawn from the urn is known with certainty.

The second triangle with a dot in the middle represents an urn of known composition, having as many balls of each color. In this case there are objective reasons to represent the color of the draw with a probability distribution.

The third triangle, where $\mathcal{C}$ is a horizontal segment, represents Ellsberg's situation where the subject only knows that one third of balls are red, but the proportion of black and yellow is unknown.

We will now read the line of triangles starting from the other end, right side. The rightmost triangle represents an urn about which we know only that it contains no *Yellow* balls. This illustrates the important class of constraint-based methods to define $\mathcal{C}$. Simply rule out from Ω impossible futures, to retain a subset $V \subset \Omega$ of all equally possible futures. Then $\mathcal{C}$ can be simply taken as the set of all probability distributions with support V.

Constraint-based methods sort futures in two crisp possibility/impossibility categories. This extreme manicheism is relaxed with possibility theory. Classically, the set

V of equally and totally possible futures can be identified with its indicator function $\pi_V : \Omega \to \{0,1\}$, defined by $\pi_V(\omega) = 1$ if and only if $\omega \in V$. Possibility theory Zadeh [1978] makes V a fuzzy set, by identifying it with a function $\pi : \Omega \to [0,1]$ that can take all the intermediate values between 0 and 1.

This possibility distribution π is normalized so that its maximum is 1. This is closely related to earlier ideas by Shackle [1952], who would interpret $1-\pi(\omega)$ as a subjective degree of surprise to be experienced if ω occurs.

The set $\mathcal{C}$ can be defined as the set of probabilities bounded above by $\overline{p}_\pi$ in the following way:

$$\mathcal{C} = \{p \,|\, \forall S,\, p(S) \leq \max_{\omega \in S} \pi(\omega)\} \tag{1}$$

This gives the key to the penultimate triangle. On the face of it, it represent an urn containing “no more than 70 percent Black balls and no more than 60 percent *Yellow* balls”. Formally, this is the possibility distribution $\pi = (1, 0.7, 0.6)$.

The degree of possibility corresponds to the maximum allowed probability. Since this later variable is particularly important, we denote $\overline{p}(S)$ the upper probability of a scenario set, it can be defined for any $\mathcal{C}$ and will also be called plausibility:

$$\overline{p}(S) = \max_{p \in \mathcal{C}} p(S) \tag{2}$$

Moving back to the ante-penultimate (fifth) triangle, the probability set there represents the information “the urn do contains balls of all three colors, biased a little toward *Red* and against *Yellow*”. The fourth triangle add to this information that, we know there are more Black balls than Yellow balls in the urn. This rules out the points corresponding to probabilities such that $p(Yellow) > p(Black)$. In this way, the sixth, fifth and fourth triangles together illustrate the idea that the giving more and more information is the same as restricting $\mathcal{C}$. Knowing permits, as Ellsberg wrote, to rule out a number of distributions.

To represent more complicated shapes of $\mathcal{C}$, powerful mathematical tools are available. For example, the ante-penultimate triangle can be represented in two ways. One is by defining upper and lower probabilities for each ω. The other is called Dempster-Shafer theory, which controls $\overline{p}$ using a probability distribution m defined on subsets of Ω, and then defines $\overline{p}_m(S) = \sum_{S \cap E \neq \emptyset} m(E)$. In the fourth triangle, $\mathcal{C}$ can be defined with linear constraints. In the third triangle, $\mathcal{C}$ is defined by giving its summits.

While there are many methods to define $\mathcal{C}$, not all are equally convenient. Pending the development of software to deal with belief networks and other imprecise probability formalisms, possibility theory seems today the best available tool to quantify uncertainty about scenarios, see for example Young [2001]. This is why this paper will, after discussing rules and principles at the more general level of probability sets, use possibility theory to present the illustrative example.

2.3 Interpretation and motivation.

The problem we are interested in represents a situation of one-way communication from an analyst to a decision-maker. The analyst knows a lot of detailed information (represented by $\mathcal{C}$), but he has limited bandwidth to transmit to the decision-maker. How should he pick S to best control the loss of information?

Formally, I take $(\Omega, \mathcal{C})$ as given, and discuss how to determine S. The goal is not to argue for a normatively best method, but to offer a set of sensible and practical rules to help justify the choice of specific scenarios.

From a classical signal theory point of view, one would seek to minimize the unavoidable information loss occurring in the communication. The difficulty is that imprecise probabilities explicitly recognize that ignorance (and therefore information) is not unidimensional [Smithson, 1988].

Consider for example the classical Entropy definition of information. [Klir, 1999] remarked that a good measure the information content of $\mathcal{C}$ is Aggregate Uncertainty, defined as the maximum of the entropy reached by probability distributions $p \in \mathcal{C}$:

$$\overline{H}(\mathcal{C}) = \max_{p \in \mathcal{C}} - \sum_{i=1}^{|S|} p_i \log_2 p_i \tag{3}$$

Using our lineup of Ellsberg's urns, it is easy to show that this measure does not capture everything. Changing from the first to the second triangle increases the upper entropy since it goes from zero (for a full information) to about $\log_2 3$ (for equiprobability among three alternatives). But changing from the second to the third triangle does not increase $\overline{H}$ since it is already at its maximum. Yet the larger $\mathcal{C}$, the less available information about the urn's composition.

This is because upper entropy is only related to the position of $\mathcal{C}$, it measures how close p can be to the center. It does not inform about the shape or the size of $\mathcal{C}$. No single number can summarize both the size and the position of a geometrical figure at the same time. This intuitively explains why it would be difficult to determine S by simply minimizing information loss using a single real-valued measure of uncertainty.

What I propose instead is a toolbox of mathematical principles to complement the many existing informal techniques of Scenario analysis.

3 Principles

In this section, I discuss three rules to determine S: scenarios should be all equally valid, S should be plausible, and it should describe a wide range of possible futures. The number of scenarios is not fixed, it can varies between 2 to 4 or more. These three rules together define a lexicographic order, allowing to recognize a good S from a bad one. They are presented the most important first.

3.1 No scenario more probable than another

Because all probabilities in $\mathcal{C}$ are equally possible, one can only say that a future ω_1 is more probable than ω_2 if and only if $p(\omega_1) > p(\omega_2)$ for all p in $\mathcal{C}$. For example, in Ellsberg's urn no color is more probable than any other[1].

[1] But if one knew in addition that there were at least 31 yellow balls, then yellow would be more probable than red.

I take as principle one that according to $\mathcal{C}$, no future in S should be more probable than any other. This formal property is crucial so that no scenario should be dismissed out of hand as less probable than another. Note that this principle is not about indifference or keeping 'all scenarios more or less equally plausible'. It states that uncertainty is so large that no scenario can be preferred to another[2].

Remark that if equiprobability (that is, the middle of the triangle) is in the interior of $\mathcal{C}$, then no element of S is more probable than another[3]. This suggest a way to operationalize the principle:

Rule 1 (No preference) *Choose S such that equiprobability on S belongs to the interior of $\mathcal{C}$.*

When S is presented without information about the relative likelihood of the different scenarios, a Bayesian analyst is likely to use a so-called uninformative, that is uniform, probability distribution. The advantage of this method is that at least the Bayesian analyst is not led into using an unreasonable probability distribution, where unreasonable means ruled out by the analysis that led to $\mathcal{C}$[4].

An analyst using the imprecise probabilities would use a set of probability distributions on S: the subset $\mathcal{M}$ of all $p \in \mathcal{C}$ those support is S. For him Rule 1 says that S maximizes aggregate uncertainty of the $\mathcal{M}$ it induces. The other formulation is simpler, though.

3.2 Maximizing plausibility

The second principle focuses on getting the most likely scenarios on board. To this end, I propose to maximize the upper probability of a subset $S \subset \Omega$, defined as the maximum of $p(S)$ for all p in $\mathcal{C}$ by Equation 2.

Rule 2 (Maximum plausibility) *Choose S to maximize $\overline{p}(S)$.*

This principle needs more discussion, it is related to the various ways of defining $(\Omega, \mathcal{C})$. In practice, it connects with the question of whether to include a Business-as-Usual scenario.

Oftentimes, having a business-as-usual scenario is not desirable because it would become an 'official future' exclusive point of focus, thus defeating the very purpose of scenario analysis. In this case using constraint-based methods to define $\mathcal{C}$ helps since all possible futures are equally and totally plausible. In this situation the first principle bans impossible futures from S, so maximum plausibility is automatically satisfied. With constraints-based methods, the second principle is moot.

But having a clear business-as-usual scenario may be desirable. Demonstrating knowledge of conventional assumptions helps to establish the credibility of the analyst (who may be an outsider consultant), for example. Then $(\Omega, \mathcal{C})$ should be set up in a way that there is only one ω^* such that $\overline{p}(\{\omega^*\}) = 1$. It is then necessary and sufficient

[2] Incomparability (dropping the completeness axiom) is the key difference between decision-making with imprecise probabilities and non-expected utility theories that came before it.

[3] The converse is false in general, consider for example an Ellsberg urn with 20 red balls and 70 black or yellow balls.

to include ω^* in S to maximize the plausibility of the scenario set. This route will be illustrated in the second part of this paper, using possibility theory to define $\mathcal{C}$.

3.3 Maximizing contrast

The third criteria for a good scenario set is that the futures are well contrasted, so that it represents the diversity of possible futures. Consider a variable of interest $J(\omega)$, interpreted as a performance criteria for the system being analyzed. For business scenarios J may be profits, for public policy scenarios J may be social welfare, for environmental economics scenarios J may be the level of pollution.

For any scenario set, I propose to adopt the range $[\inf_{\omega \in S} J(\omega), \max_{\omega \in S} J(\omega)]$ as an indicator of the diversity of the scenario set that should be maximized:

Rule 3 (Diversity) *Choose S to maximize the range of the variable of interest.*

This principle is somewhat ambiguous, since it is not always possible to say that an interval is larger than another (both may be of the same size). But that ambiguity has not been a problem in the pilot study presented next. This is because, while no structure is assumed on Ω in theory, in practice a scenario corresponds to a n-uple $(x_1, \ldots, x_n)$, each argument x_i being a numerical parameter. And J is monotonous in each parameter.

In this practical case, this rule lead to choose two extremes, one 'low' scenario that has all the parameters minimizing J, the other maximizing it. This rule however is subordinated to the first principle of no preference: the plausibility $\overline{(}\{\omega\})$ of these extreme scenarios should be greater than $1/|S|$, where $|S|$ denotes the number of elements of S. So while these are extreme, their plausibility remains at significant levels.

Combining the three rules together, this theoretical study of scenario-making suggests these practical formal approaches to scenario-making:

- If some symmetry is desired, use constraint-based methods to rule out unrealistic futures. Define a primary performance variable, and pick two realistic futures that make it extremal. If four scenarios are desired, use a second different performance variable.

- If a Business-as-usual scenario is warranted, use a possibility function $\pi(\omega)$ that bounds the upper probability of scenarios to represent the information. Then build a set of three scenarios. The less surprising scenario has possibility level 1, the two other have possibility level 1/3 and correspond to low and high future performance.

- Another reason not to have a business-as-usual scenario may be dissonance between different schools of thought, or the possibility of multiple equilibriums. Evidence theory [Shafer, 1976] is a natural framework to manipulate $\mathcal{C}$ in this case.

Given that today possibility theory is the easiest way to represent $\mathcal{C}$, the following section uses it to illustrate these rules with a mock study of Global Warming scenarios.

4 Illustration: an hypothetical application to global warming scenarios

4.1 Another scenarios versus forecasts controversy

The Intergovernmental Panel on Climate Change [IPCC, 2000] recently elaborated long-term greenhouse gases emissions scenarios, in part to drive global ocean-atmosphere general circulation models, and ultimately to assess the urgency of action to prevent the risk of climatic change.

Using these scenarios led the IPCC to report a range of global warming over the next century from 1.4 to 5.8 °C, without being able to report any likelihood considerations. This turned out to be controversial, as it dramatically revised the top-range value which was previously 3.5 °C. Yet some combinations of values which lead to high emissions, such as high per capita income growth and high population growth, appear less likely than other combinations. The debates then fell into usual controversy between the makers and the users of scenarios:

- Schneider [2001], as well as Reilly et al. [2001] argued that the absence of any probability assignment would lead to confusion, as users select arbitrary scenarios or assume equiprobability. As a remedy, Reilly et al. estimated that the 90% confidence limits were 1.1 to 4.5 °C, while Wigley and Raper [2001] found 1.7 to 4.9 °C for the same 1990 to 2100 warming.

- Grübler and Nakicenovic [2001] and Allen et al. [2001] took the opposite side by arguing that good scientific arguments preclude determining 'probabilities' or the likelihood that future events will occur. They explained why it was the unanimous view of the IPCC report lead authors that no method of assigning probabilities to a 100-year climate forecast was sufficiently widely accepted and documented to pass the review process. They underlined the difficulty of assigning reliable probabilities to socioeconomic trends in the latter half of the 21st century, and the difficulty of obtaining consensus range for quantiles like climate sensitivity, and the possibility of a nonlinear geophysical response.

The method described in this paper could be applied to solve this controversy. It would require an in-depth analysis of the determinants of greenhouse gases emissions in the different areas of the world, coupled with a synthesis of scientific information about global warming mechanism. One would then be able to derive a good $(\Omega, \mathcal{C})$ and proceed to derive convincing possibility levels for future global warming. My goal here is much more modest, to illustrate with a pilot study how such a deeper exercise would play out.

The variable of interest here is global warming in year 2100 denoted ΔT_{2100} in degree Celsius. Two parameters ($n = 2$) describe the futures. Parameter $[CO_2]_{2100}$ is the global atmospheric concentration of CO_2 in 2100, measured in parts per million in volume (ppmv). Parameter $\Delta T_{2\times CO_2}$ is climate sensitivity, that is the warming that would occur in the long run if the atmospheric concentration of CO_2 was stabilized at twice the preindustrial concentration level (two times 275 ppmv approximately).

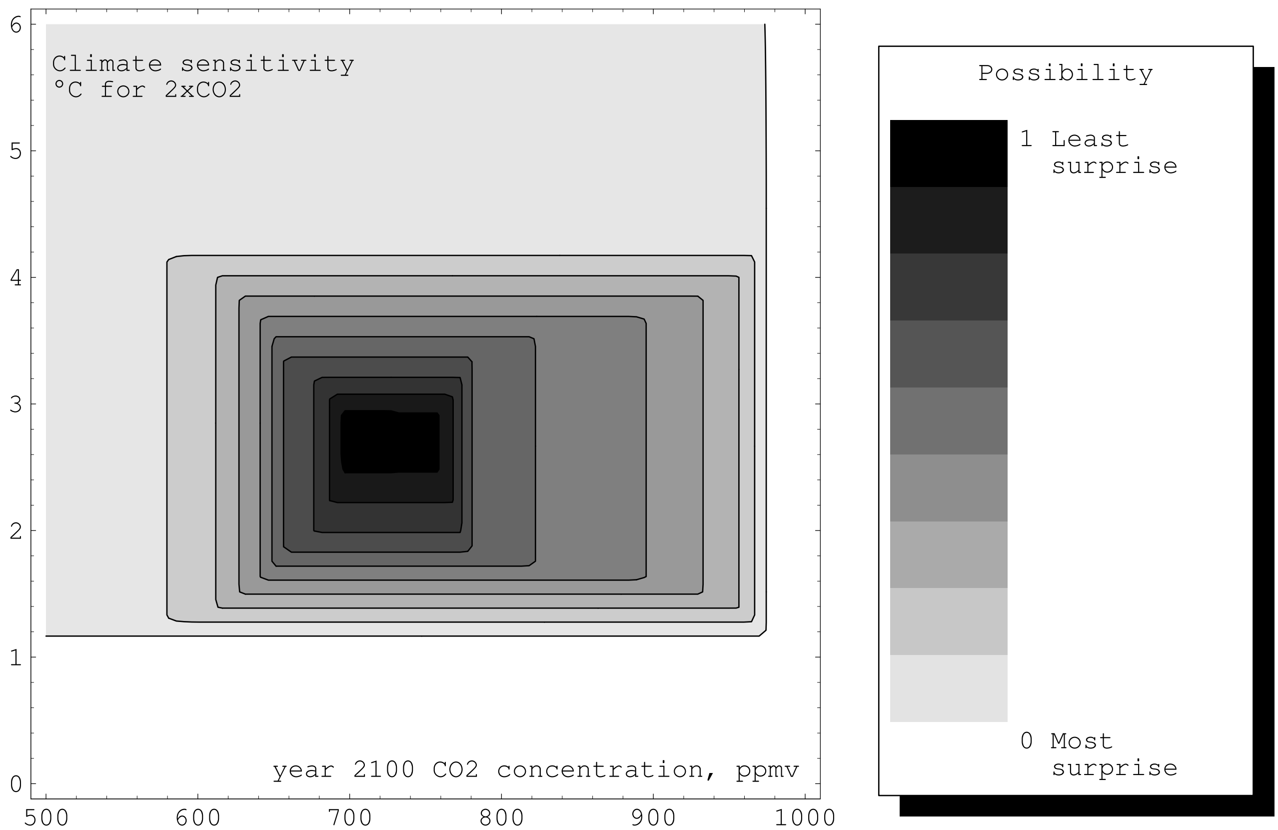


Figure 1: Contour lines of a possibility function.

Because it takes a long time to realize this long-term equilibrium, I assumed that the variable of interest is determined as follows:

$$\Delta T_{2100} = \frac{[CO_2]_{2100} - 275}{275} \times \frac{\Delta T_{2\times CO_2}}{2} \tag{4}$$

Admittedly this is the crudest model of global warming. Results will be sensitive to the gross assumption that half of the long term warming effect is realized instantaneously. Numerical results should be taken with no more confidence than the data and the model warrants, that is none at all. This analysis is done solely to demonstrate the value of using imprecise probability-based methods.

4.2 Results

The contour plot Figure 1 illustrates the joint possibility $\pi([CO_2]_{2100}, \Delta T_{2\times CO_2})$ determining $\mathcal{C}$. The horizontal axis represents levels of atmospheric CO_2 concentrations in

	Low	Middle	High
CO_2 concentration in 2100, ppmv	618	709	944
Climate sensitivity at $2\times CO_2$, °C	1.43	2.85	3.95
Global Warming in 2100, °C	0.9	2.3	4.8
Possibility	0.34	1	0.34

Table 1: The set of three possible futures in the pilot study. Numeric values are purely illustrative.

2100. The vertical axis represents the climate sensitivity. The figure shows contour lines of the possibility function. For example, the possibility that (concentration will be between 660 and 840 parts per million AND climate sensitivity between 1.6 and 3.5) is greater than 0.6.

In a predictive model, one might assume that if climate sensitivity turns out on the very high side, then serious actions will be taken to reduce pollution. This pilot study, however, is not interested in prediction but in global warming if nothing is done to prevent it. This is why it was assumed the two variables do not interact: the possibility is the minimum of the possibilities $\pi([CO_2]_{2100})$ and $\pi(\Delta T_{2\times CO_2})$. Contour lines Figure 1 are rectangles, it is only due to numerical approximations that the angles do not look sharply right.

This leaves to explain how the possibility level of each variable was determined. I used a different method for each:

The possibility distribution of climate sensitivity was determined using data from Morgan and Keith [1995]. This dataset collects the subjective probability distribution of sixteen leading world experts on climate sensitivity. The experts' opinions were fusioned using the method described in Smets [2000].

The possibility of future CO_2 concentration level was derived using Joslyn [1997] theory of possibilistic histograms applied to the IPCC SRES database by Morita and Lee [2000], and then converting global cumulative carbon emissions over the century to atmospheric concentration with a linear model.

There are other methods than fusioning experts judgments and combining model results. In a more realistic exercise, one would presumably not use frequency information from integrated assessment models, but rather work with more than two parameters to be able to use specific information from disciplinary fields.

Figure 2 illustrates how the set of three possible futures, given Table 1, was determined. The two rectangles in figure show the contour lines at the 0.34 and 0.95 possibility levels.

Following the first rule, I picked all futures at a possibility level strictly above 1/3, that is inside the 0.34 contour line. Following to the second rule, I picked the least surprising future as the one with possibility 1, located inside the 0.95 possibility contour.

The third rule suggest to pick the two extreme scenarios that maximize and minimize global warming. To visualize this, the figure shows the iso-warming curves: branches of hyperbolas since Equation 4 is a product. Larger values of the parameters represent a larger global warming in 2100, so in this family of curves warming

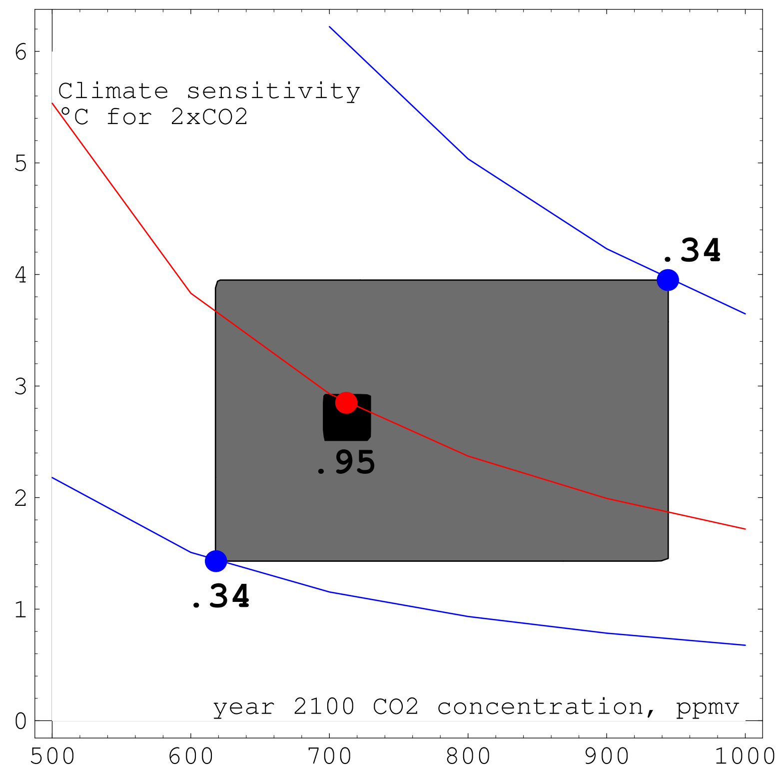


Figure 2: Set of three possible futures. The plausibility is maximal because the least surprising scenario is part of the set. The other two points give Low and High levels of global warming. But no scenario is more probable than another, the uniform probability (1/3, 1/3, 1/3) is not ruled out.

increase when moving in the north-east direction. Clearly, this family of curves is oriented so that the top right and the bottom left corners of the $\pi \geq 0.34$ rectangle realize the extreme values.

5 Concluding remarks

This paper propose to bridge the gap between forecasts, which are probabilized, and scenarios which are not, by using set of probabilities as the fundamental tool to represent information. Such sets can be defined using possibility distributions or constraint-based methods, depending on the desirability to produce a Business-as-usual scenario. I offer three formal rules to build sets of possible futures: no scenario should be more probable than another, the set should be maximally plausible, and it should describe a maximal range for the variable of interest. These rules have been illustrated with an hypothetical study of global warming by the end of the next century.

In conclusion, I would compare the techniques developed here with the rules of Linear Perspective in painting and drawing, an invention attributed to the Italian renaissance artist Filippo Brunelleschi (1377–1466). Scenario analysis may be an Art or a Science, imprecise probabilities are just but a technical tool. May this toolbox be a useful to practitioners and consultants concerned with the strategic choices under uncertainty.

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